1) a $\quad \vec{A}\cdot(\vec{B}\times\vec{C}) = \sum_i A_i (\vec{B}\times\vec{C})_i = \sum_i A_i \left(\sum_{jk} \varepsilon_{ijk} B_j C_k\right) = \boxed{\sum_{ijk} \varepsilon_{ijk} A_i B_j C_k}$ الف)

$$(\vec{A}\times\vec{B})\cdot\vec{C} = \sum_i (\vec{A}\times\vec{B})_i C_i = \sum_i \sum_{jk} \varepsilon_{ijk} A_j B_k C_i = \sum_{ijk} -\varepsilon_{jik} A_i B_k C_j$$

$$= \sum_{ijk} + \varepsilon_{jki} A_i B_j C_k = \boxed{\sum_{ijk} \varepsilon_{ijk} A_i B_j C_k}$$

ب)

$$r = k(1+\cos\theta) \longrightarrow \dot{r} = -k\sin\theta\,\dot\theta$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot\theta\hat{e}_\theta \Rightarrow |\vec{v}|^2 = \dot{r}^2 + r^2\dot\theta^2 = k^2\sin^2\theta\,\dot\theta^2 + k^2(1+\cos\theta)^2\dot\theta^2$$

$$v = \dot\theta\sqrt{k^2\sin^2\theta + k^2 + k^2\cos^2\theta + 2k^2\cos\theta} = \dot\theta\sqrt{2k^2(1+\cos\theta)}$$

$$\Rightarrow \boxed{v = \dot\theta\sqrt{2kr}}$$

۲

$$F = mkvx = m\frac{dv}{dt} = m\frac{dv}{dx}\cdot\frac{dx}{dt} \Rightarrow \cancel{m}k\cancel{v}x = \cancel{m}\frac{dv}{dx}\cdot\cancel{v}$$

$$\int_{v_0}^{v} dv = k\int_0^x x\,dx \Rightarrow v = v_0 + \frac{kx^2}{2} \Rightarrow \frac{dx}{dt} = v_0 + \frac{k}{2}x^2$$

$$\Rightarrow \int_0^x \frac{dx}{x^2 + \frac{2v_0}{k}} = \frac{k}{2}\int_0^t dt \Rightarrow \sqrt{\frac{k}{2v_0}}\tan^{-1}\frac{x}{\sqrt{\frac{2v_0}{k}}} = \frac{k}{2}t \Rightarrow \boxed{x = \sqrt{\frac{2v_0}{k}}\tan\left(\sqrt{\frac{kv_0}{2}}\,t\right)}$$

۳

$$F(x) = x - x^3$$

تعادل: $\frac{dU}{dx} = -F(x) = 0 \Rightarrow x(1-x^2) = 0 \Rightarrow \boxed{x = 0, \pm 1}$ نقاط تعادل

$$\frac{d^2U}{dx^2} = -\frac{dF}{dx} = -1 + 3x^2 \Longrightarrow \begin{cases} x = 0 \Rightarrow U'' = -1 < 0 & \text{تعادل ناپایدار} \\ x = \pm 1 \Rightarrow U'' = +2 > 0 & \text{تعادل پایدار} \end{cases}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''}{m}} = \sqrt{\frac{2}{1}} = \sqrt{2}$$ فرکانس نوسانات حول $x=\pm 1$

$$E = T + U = \frac{1}{2}m\dot{x}^2 + U(x) = C_1 + C_2 = \text{ثابت}$$

$$F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx} = x - x^3 \Rightarrow m\int \dot{x}\,d\dot{x} = \int (x - x^3)\,dx \Rightarrow \frac{m\dot{x}^2}{2} = \frac{x^2}{2} - \frac{x^4}{4} + C_1$$

$$F(x) = -\frac{dU}{dx} \Rightarrow \int dU = -\int F(x)\,dx = -\int (x - x^3)\,dx = -\frac{x^2}{2} + \frac{x^4}{4} + C_2$$

٤) الف)

$$x = D\cos(\omega_1 t - \delta) \quad , \quad D_{(t)} = Ae^{-\beta t}$$

$$D(t=3\tau) = D(t=0)/e \Rightarrow Ae^{-\beta 3\tau} = \frac{Ae^{0}}{e} \Rightarrow \boxed{3\beta\tau = 1} \quad , \quad \tau = \frac{2\pi}{\omega_1}$$

$$\Rightarrow \boxed{\beta = \frac{\omega_1}{6\pi}} \quad , \quad \beta^2 = \omega_0^2 - \omega_1^2 \Rightarrow \frac{\omega_1^2}{\omega_0^2} = \frac{1}{1+\left(\frac{1}{6\pi}\right)^2}$$

$$\frac{\omega_1}{\omega_0} = \frac{1}{\sqrt{1+\left(\frac{1}{6\pi}\right)^2}} \simeq 1 - \frac{1}{2}\left(\frac{1}{6\pi}\right)^2 = 1 - \frac{1}{72\pi^2} \simeq 1 - \frac{1}{700} \Rightarrow \frac{\omega_1}{\omega_0} \simeq 1 - 0.15\%$$

٤-ب) از معادله پواسون استفاده می‌کنیم:

$$\nabla^2\varphi = 4\pi G\rho_{(r)}$$

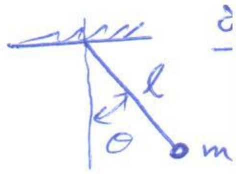
$$\vec{g} = -\vec{\nabla}\varphi = cte \Rightarrow -\frac{\partial\varphi}{\partial r} = g = \text{ثابت}$$

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-gr^2\right) = -2g/r$$

معادله پواسون $\nabla^2\varphi = 4\pi G\rho \Rightarrow \rho = -\frac{2g}{4\pi G} \times \frac{1}{r} \Rightarrow \boxed{\rho_{(r)} = \frac{-g}{2\pi G} \times \frac{1}{r}}$

ج)

$$\dot{l} = \frac{dl}{dt} = -\alpha \quad ,$$



$$T = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2\right) = \frac{1}{2}m\left(\alpha^2 + l^2\dot{\theta}^2\right)$$

$U = -mgl\cos\theta$ (نقطه آویز سطح پتانسیل صفر)

$$L = T - U \Rightarrow \boxed{L_{(\theta,\dot{\theta})} = \frac{1}{2}m\alpha^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta}$$ تابع لاگرانژی

$$P_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{P_\theta}{ml^2}$$

$$H = H(P_\theta,\theta) = P_\theta\dot{\theta} - L = \frac{P_\theta^2}{ml^2} - \frac{1}{2}m\alpha^2 - \frac{1}{2}ml^2\frac{P_\theta^2}{(ml^2)^2} - mgl\cos\theta$$

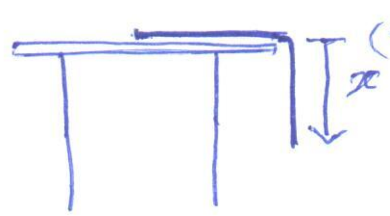
$$\boxed{H = -\frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{P_\theta^2}{ml^2} - mgl\cos\theta}$$ تابع هامیلتونی

$$\boxed{E = T + U = \frac{1}{2}m\alpha^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta}$$ انرژی کل

— انرژی کل در سیستم ثابت نیست چرا که $l = l(t)$ تابع زمان است.

— هامیلتونی و انرژی کل برابر نیستند $E \neq H$ (به علامت $\frac{1}{2}m\alpha^2$ توجه کنید) چرا که تبدیل تابع زمان است

۹ الف)



طول قسمت آویزان شده $= x$ ⟸ جرم قسمت آویزان شده $= \frac{mx}{L}$

محل مرکز جرم قسمت آویزان شده $= \frac{x}{2}$ ⟸ $U = -mg\frac{x^2}{2L}$

چون ریسمان بدون پیچش است هر قسمت آن با سرعت ثابت $\dot{x}$ حرکت می‌کند

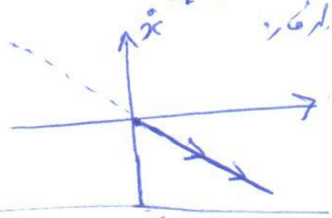
$$T = \frac{1}{2}m\dot{x}^2$$

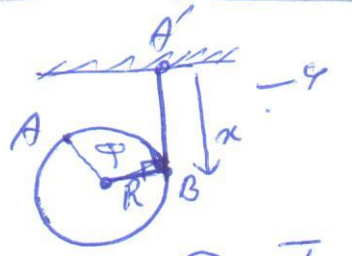
$$E = T + U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\frac{mg}{L}$$

$$\dot{x}^2 = \frac{g}{L}x^2 \Rightarrow \dot{x}$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + mg\frac{x^2}{2L}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow \frac{mgx}{L} - m\ddot{x} = 0 \Rightarrow \boxed{\ddot{x} = \frac{gx}{L}}$$





$$\widehat{AB} = A'B$$

$$U = -mgx$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\varphi}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{5}mR^2\dot{\varphi}^2$$

چون نیروی قیدی خواسته شده قید $f(x,\varphi) = x - R\varphi = 0$ را فعلاً وارد می‌کنیم.

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{5}mR^2\dot{\varphi}^2 + mgx$$

$$\begin{cases} \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial f}{\partial x} = 0 \\ \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} + \lambda\frac{\partial f}{\partial \varphi} = 0 \end{cases} \Rightarrow \begin{cases} mg - m\ddot{x} + \lambda = 0 \\ 0 - \frac{2}{5}mR^2\ddot{\varphi} - R\lambda = 0 \Rightarrow \frac{2}{5}mR\ddot{\varphi} \end{cases}$$

قید $x = R\varphi \Rightarrow \ddot{x} = R\ddot{\varphi} \Rightarrow \left.\begin{cases} mg - m\ddot{x} + \lambda = 0 \\ m\ddot{x} + \frac{5}{2}\lambda = 0 \end{cases}\right\} \Rightarrow \boxed{\lambda = -\frac{2}{7}mg}$

شتاب حرکت $\boxed{\ddot{x} = \frac{5}{7}g}$

نیروی قیدی $Q_x = \lambda\frac{\partial f}{\partial x} = -\frac{2}{7}mg$ نیرویی که نخ به قرقره وارد می‌کند $\Rightarrow$ نیروی نخ $= -\frac{2}{7}m$

الف) ۷

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu r^2}{l^2}F(r)$$

$$r=k\theta^2 \rightarrow \frac{d}{d\theta}\left(\frac{1}{r}\right)=\frac{-2}{k\theta^3} \Rightarrow \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)=\frac{6}{k\theta^4}=\frac{6k}{r^2}$$

$$\Rightarrow \frac{6k}{r^2}+\frac{1}{r^2}=\frac{-\mu r^2}{l^2}F(r) \Rightarrow \boxed{F(r)=\frac{-l^2}{\mu}\left(\frac{6k}{r^4}+\frac{1}{r^3}\right)}$$

$$\frac{d\theta}{dt}=\dot{\theta}=\frac{l}{\mu r^2}=\frac{l}{\mu k^2\theta^4} \Rightarrow \int\theta^4 d\theta=\int\frac{l}{\mu k^2}dt \Rightarrow \frac{\theta^5}{5}=\frac{l}{\mu k^2}t+C'$$

$$\boxed{\theta_{(t)}=\sqrt[5]{\frac{5l}{\mu k^2}t+C}}$$

ب-۷

$$F(r)=\frac{a}{r^2}+\frac{b}{r^4} \Rightarrow U(r)=-\int F\,dr \Rightarrow \underset{\text{موثر}}{V(r)}=U(r)+\frac{l^2}{2\mu r^2}$$

شرط حرکت دایره‌ای $\quad r=cte \Rightarrow \dot{r}=0 \Rightarrow \frac{dV}{dr}=0=\frac{dU}{dr}+\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right)$

$$\Rightarrow -F(r)-\frac{l^2}{\mu r^3}=0 \Rightarrow \frac{a}{r^2}+\frac{b}{r^4}+\frac{l^2}{\mu r^3}=0 \Rightarrow \boxed{l^2=-\mu\left(ar+\frac{b}{r}\right)}$$

شرط پایداری : $\quad \left.\frac{d^2V}{dr^2}\right|_{r_0}>0 \Rightarrow -\frac{dF}{dr}+\frac{3l^2}{\mu r^4}>0 \Rightarrow \frac{+2a}{r^3}+\frac{4b}{r^5}+\frac{3}{\mu r^4}(-\mu(a$

$$\Rightarrow \frac{-a}{r^3}+\frac{b}{r^5}>0 \Rightarrow \boxed{b>ar^2}$$ شرط پایداری

قسمت آخر روش دوم :

$$\frac{F'(\rho)}{F(\rho)}+\frac{3}{\rho}>0 \Rightarrow \frac{\frac{-2a}{\rho^3}-\frac{4b}{\rho^5}}{\frac{a}{\rho^2}+\frac{b}{\rho^4}}+\frac{3}{\rho}>0 \Rightarrow \frac{-2a\rho^2-4b}{a\rho^3+b\rho}+\frac{3}{\rho}>0$$

$$\Rightarrow \frac{a\rho^3-b\rho}{\ldots}>0 \longrightarrow \boxed{a\rho^2<b}$$