Computational Physics, Project #3

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2D Laplace Equation

Several physical quantities satisfy the Laplace equation $\nabla^2\phi(\mathbf{r}) = 0$. For instance, electric potential in the case of no charge and in two dimensions satisfies

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0. \tag{1}$$

If Dirichlet boundary conditions

$$\begin{cases} V(x,0) = V^0, \\ V(x,b) = V(0,y) = V(a,y) = 0 \end{cases} \tag{2}$$

apply to a rectangular domain $0 \le x \le a = 1$ and $0 \le y \le b$, there is an analytic solution

$$V(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{\sin \pi x}{\sinh \pi y}\right) \tag{3}$$

for $b \to \infty$.

In this project, you will solve Eq. (1) with the finite difference method. The domain is defined as $0 \le x \le 1$ and $1 \le y \le 2$ with boundary conditions

$$\begin{cases} V(0,y) = V(1,y) = 0, \\ V(x,1) = \frac{2}{\pi}\tan^{-1}\left(\frac{\sin \pi x}{\sinh \pi}\right), \\ V(x,2) = \frac{2}{\pi}\tan^{-1}\left(\frac{\sin \pi x}{\sinh 2\pi}\right) \end{cases} \tag{4}$$

The exact solution is given by Eq. (3) and plotted in Fig. 1. It is used to determine the error of the numerical solution

$$||e|| = \frac{1}{N}\sqrt{\sum_{i,j}\left(V_{i,j}^{\text{numeric}} - V_{i,j}^{\text{exact}}\right)^2}. \tag{5}$$

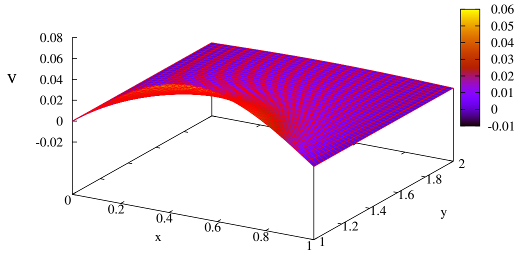

Figure 1: Exact solution of the equation on the region $x \in [0,1], y \in [1,2]$

1 Discretization

In a numerical solution, $V(x, y)$ is available only on a set of grid points. For a 2D mesh, the potential at (x, y) is represented as $V_{i,j}$ where the grid points are determined by $(x = ih_x, y = jh_y)$ with $i = 0, \cdots, N_x$ and $= 0, \cdots, N_y$. For our square domain of length 1, let use the same number of grid points in both directions $N_x = N_y = N$. The grid spacing of this uniform mesh is $h = 1/N$.

1. Using Taylor's expansion, show that the discretized version of the Laplace equation (Eq. 1) on a 2D mesh is given by
$$V_{i,j} = \widetilde{V}_{i,j}^{+} + O(h^2)$$
where the 'cross' average is given by
$$\widetilde{V}_{i,j}^{+} = \frac{1}{4}\left(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}\right). \tag{6}$$

2. Do the same task for the 'corner' average
$$\widetilde{V}_{i,j}^{\times} = \frac{1}{4}\left(V_{i+1,j+1} + V_{i-1,j+1} + V_{i+1,j-1} + V_{i-1,j-1}\right) \tag{7}$$

3. Do the same task for the '8-point' average
$$\widetilde{V}_{i,j}^{\square} = \frac{4}{5}V_{i,j}^{+} + \frac{1}{5}V_{i,j}^{\times}. \tag{8}$$

2 Relaxation method

The relaxation method is a simple method to solve electrostatic problems. In this iterative method, the new value of the potential at each grid point is set to the average of the current values over its neighbors. The iteration stops when the potential on all grid points do not vary any more up to some criterion.

The simple relaxation method called the **Gauss-Seidel** method
$$V_{i,j}^{\text{new}} = \widetilde{V}_{i,j} \tag{9}$$
can be replaced by the **successive over-relaxation** method
$$V_{i,j}^{\text{new}} = \omega\widetilde{V}_{i,j} + (1-\omega)V_{i,j} \tag{10}$$
to speed up the convergence to the solution. The optimum relaxation factor in our case is $\omega = 2/(1 + \sin \pi h)$. Note that the Gauss-Seidel relaxation corresponds to $\omega = 1$.

3 Implementation

1. Write your code to perform successive over relaxation using the 'cross' average. Use random values as initial guess of the potential on grid points. By trail and error, find a reasonable criterion for stopping the iteration.

2. Calculate error analysis using Eq. (5). Show $||e||$ as a function of h in a log-log plot and determine the order of the error.

3. Repeat steps 1 and 2 with the 'corner' average and compare the errors.

4. Repeat steps 1 and 2 with the '8-point' average. Does eight points for averaging improve the accuracy?

4 Curved Boundaries

The boundaries in the above problem are perfectly described by the square mesh. However, if there is a curved boundary, then errors related to the description of the boundaries come into play. In the following you will first see this effect and then by using a different system of coordinates will fix the problem related to the boundaries.

Consider the electric potential within two concentric conducting cylinders of radii $r_1 < r_2$ with very long lengths. As boundary conditions, use $V = 1$ on the inner cylinder and $V = 0$ on the outer one. If $r_1 = 1/2$ and $r_2 = 1$ the analytic solution is

$$V(x, y) = -\left(\frac{1}{\ln 2}\right) \ln \sqrt{x^2 + y^2} \tag{11}$$

4.1 Implementation

1. Adapt your code of the previous part to this geometry. Calculate $V(x, y)$ for the region $1/2 < \sqrt{x^2 + y^2} < 1$.

2. Do error analysis for the three different averages ('corner', 'cross', and '8-point'). Is the error scales as before? Why?

4.2 Laplace Equation in Polar Coordinates

If we use polar coordinates system, the error due to curved boundaries will disappear in the present problem. Because of the cylindrical symmetry, the potential is only a function of r (i.e. we have a 1D problem in the polar system).

1. Discrete the Laplace equation in *polar coordinates*.

2. Implement the derived discrete equation, and determine the error order with respect to the grid spacing.