Date: 1401/03/24

In the name of God

Department of Physics, Shahid Beheshti University

STOCHASTIC PROCESSES

Final exam

(Time allowed: 3 hours)

NOTE: All questions must be answered. Please write the answer of each question in separate sheet.

1. Suppose that a brownian particle is living in a potential well with time independent potential as $\Phi(x) = \alpha x^2/2$ and it is affected by noise as $\eta(t)$ with $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = 2g\delta_D(t-t')$. According to the evolution equation for position as:

$$\frac{dx}{dt} = -\frac{d\Phi}{\gamma dx} + \eta(t)$$

here γ is a constant.

(a) Write the Fokker-Planck equation for $p(x,t)$. (5 points)

(b) Suppose that $p(x_0, t=0) = \delta_D(x_0)$ and $x_0 = 0$. Calculate $\langle x(t) \rangle$ and $\langle x(t)^2 \rangle$. (10 points)

(c) Describe your results for $t \to 0$ and $t \to \infty$. (5 points)

2. Self-similar and self-affine processes:

(a) Define Self-similar and self-affine processes and explain the difference between them. To make more sense, give an example for each mentioned processes. (10 points)

(b) What is the relation between Hurst exponent and fluctuation function derived in e.g. DFA method. (5 points)

3. First passage time: the $p(x,t)$ is the probability density of finding the particle in location x at t. According to mentioned PDF one can define the first passage time probability density function as the probability density function that a particle has first reached a given point such as x_f at exactly t_f. To this end we rely on so-called survival probability as the probability that the particle has remained at a position $x < x_f$ for all times up to t, as:

$$S(x_f, t) \equiv \int_{-\infty}^{x_f} p(x,t)dx$$

therefore, the probability of reaching the particle to x_f at t is $f(x_f, t) = -\frac{\partial S(x_f,t)}{\partial t}$ and for common brownian motion we have:

$$p(x,t) = \frac{1}{\sqrt{4\pi g^2 t}} \exp\left(-\frac{x^2}{4g^2 t}\right)$$

$$f(x_f,t) = \frac{|x_f|}{\sqrt{4\pi g^2 t^3}} \exp\left(-\frac{x_f^2}{4g^2 t}\right)$$

(a) Derive the $\langle t^n \rangle$ for the first passage time probability density function in terms of survival probability. (5 points)

(b) For $t \to 0$ and $t \to \infty$ regimes, derive $f(x_f, t)$ and explain the physical meaning of your results. (10 points)

Good luck, Movahed

بسم الله الرحمن الرحیم

پاسخنامه امتحان پایان ترم

درس فرآیندهای تصادفی

۱۴۰۱/۳/۲۴

Answer 1

a) $$\frac{dx}{dt} = -\frac{d\phi}{\gamma dx} + \eta(t) \quad , \phi = \frac{\alpha x^2}{2}$$

$$\frac{d\phi}{dx} = \alpha x \longrightarrow \begin{cases} \frac{dx}{dt} = -\frac{\alpha}{\gamma} x + \eta(t) \\ \langle \eta(t)\eta(t')\rangle = 2g\,\delta_D(t-t') \end{cases}$$

therefore $D^{(1)} = -\frac{\alpha}{\gamma} x$

$D^{(2)} = g$

5/5 points

b) $\langle x(t)\rangle = ?$

$\langle x^2(t)\rangle = ?$

by green's function method we have

$$x(t) = x_0 e^{-\frac{\alpha}{\gamma}t} + \int_0^t dt' \, e^{-\frac{\alpha}{\gamma}(t-t')} \eta(t')$$

$$\langle x(t)\rangle = x_0 e^{-\frac{\alpha}{\gamma}t} + 0 = x_0 e^{-\frac{\alpha}{\gamma}t}$$ 5/5 points

$$\langle x(t)x(t)\rangle = \left\langle \left[x_0 e^{-\frac{\alpha}{\gamma}t} + \int_0^t dt' \, e^{-\frac{\alpha}{\gamma}(t-t')} \eta(t')\right]\left[x_0 e^{-\frac{\alpha}{\gamma}t} + \int_0^t dt'' \, e^{-\frac{\alpha}{\gamma}(t-t'')} \eta(t'')\right]\right\rangle$$

$$= x_0^2 e^{-\frac{2\alpha}{\gamma}t} + \int_0^t dt' \int_0^t dt'' \, e^{-\frac{\alpha}{\gamma}[2t-t'-t'']} \underbrace{\langle \eta(t')\eta(t'')\rangle}_{2g\,\delta_D}$$

$$= x_0^2 e^{-\frac{2\alpha t}{\gamma}} + \int_0^t dt' \, 2g \, e^{-\frac{2\alpha}{\gamma}[t-t']}$$

$$= x_0^2 e^{-\frac{2\alpha t}{\gamma}} + \frac{2g\gamma}{2\alpha} e^{-\frac{2\alpha}{\gamma}(t-t')}\Big|_0^t$$

$$\langle x^2(t)\rangle = x_0^2 e^{-\frac{2\alpha t}{\gamma}} + \frac{2g\gamma}{2\alpha}\left(1 - e^{-\frac{2\alpha}{\gamma}t}\right)$$ 5/5 points

c) $\lim_{t\to 0} \langle x(t)\rangle = x_0$ at $t=0$ the particle is in initial position

$\lim_{t\to\infty} \langle x(t)\rangle = 0$ at long time the average location of particle does not change.

$$\lim_{t\to 0} \langle x^2(t)\rangle = x_0^2$$

$\lim_{t\to\infty} \langle x^2(t)\rangle = \frac{g\gamma}{\alpha} = cts$ be carefull about that the evolution equation of posit is give by Langevin Eq. instead of velocity. therefore we can not expect to obtain $\sigma_x^2 \sim t$ as usually found for standard brownian motion

5/5 points

Answer 2.

a) Self-Similar is a shape for which each part of shape are same as the whole.

fractal shape

Self-affine is a self-similar shape but in anisotropic manner

$\frac{10}{10}$ points

fractional Brownian Motion
fractional Gaussian Noise
} Samples

b) $f^2(s) \sim S^{\alpha}$, $\alpha = H$ for fGn signal.

5/5 points

Answer 3

a). $\langle t^n \rangle = \int dt\, t^n f(x_f, t)$.

$= \int_0^{\infty} dt\, t^n \left(-\frac{\partial S}{\partial t}\right)$

$t^n = u$
$-S' dt = dV$

Integration By-Part

$= -t^n S\Big|_0^{\infty} + n\int dt\, t^{n-1} S(x_f, t)$

$= 0 + n\int_0^{\infty} dt\, t^{n-1} S(x_f, t)$

5/5 points

according to Boundary Condition

b) $\lim_{t \to 0} f(x_f, t) = \lim_{t \to 0} t^{-3/2} e^{-\frac{1}{t}} = 0$

5/5 points

This means that for $t = 0$ the particle can not reach to x_f, since it has no enough time to reach to x_f

$\lim_{t \to \infty} f(x_f, t) = \lim_{t \to \infty} t^{-3/2} e^{-1/t} = 0$

5/5 points

this means that at long time the particle can reach to any given x_f and therefore The probability of first passage for long time goes to zero