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fitting formula: linear fitting with errors

suppose That in an experiment we measure (x_i, y_i) where x_i is independent parameter and y_i is dependent parameter. In General Case suppose $x_i \pm \sigma_{x_i}$ and $y_i \pm \sigma_{y_i}$

Consider we have a model as $Y_i = m x_i + C$ (Theoretical Model)

so

$$\chi^2 \equiv \sum_{i=1}^{N} \frac{[y_i - Y_i]^2}{\sigma_{x_i}^2 + \sigma_{y_i}^2}$$

(x_i, y_i)

for best value of $m = m_{best}$ and $C = C_{best}$

$$\chi^2(m_{best}, C_{best}) = \chi^2_{min}$$

$$\boxed{\left.\frac{\partial \chi^2}{\partial m}\right|_{m_{best}} = \left.\frac{\partial \chi^2}{\partial C}\right|_{C_{best}} = 0}$$

$\rightarrow -2\sum y_i x_i + 2m\sum x_i^2 + 2C\sum x_i = 0$

$\rightarrow -2\sum y_i + 2m\sum x_i + 2C = 0$

$$\begin{cases} m_{best} = \dfrac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2}, & \bar{x} = \dfrac{1}{N}\sum x_i \\ C_{best} = \bar{y} - m\bar{x}, & \bar{y} = \dfrac{1}{N}\sum y_i \end{cases}$$

or

$$m_{best} = \frac{\sum x_i y_i - (\sum x_i)(\sum y_i)}{\sum x_i^2 - (\sum x_i)^2}$$

$$c_{best} = \frac{(\sum y_i) \sum x_i^2 - \sum x_i \sum x_i y_i}{[\sum x_i^2 - (\sum x_i)^2] N}$$

Here after the most important task to do is finding errors of m_{best} and c_{best}, namely

$$\sigma_c^2 = ? \quad \text{and} \quad \sigma_m^2 = ?$$

Generally we have $\sigma_m^2 = (\sigma_m^{sys})^2 + (\sigma_m^{stat})^2$

$$(\sigma_m^{stat})^2 = \sigma_m^2 (\text{Pop}) + \sigma_m^2 (\text{int})$$

Here I am going to compute intrinsic part of error

So

$$\sigma_m^2 (\text{int}) = \left(\frac{\partial m}{\partial y_i} \sigma_{y_i}\right)^2 + \left(\frac{\partial m}{\partial x_i} \sigma_{x_i}\right)^2$$

We take into account σ_{y_i} so $\sigma_m^2 (\text{int}) = \left(\frac{\partial m}{\partial y_i} \sigma_{y_i}\right)^2$

and $\sigma_{y_i} = \sigma$

Also: $m = \frac{1}{D} \sum \xi_i y_i$, $D \equiv \sum \xi_i^2$, $\xi_i \equiv x_i - \bar{x}$

$$\sum \xi_i = 0$$

Therefore $\sigma_m^2 (\text{int}) = \frac{\xi_i^2}{D^2} \sigma_{y_i}^2 = \frac{\sigma^2}{D^2} \sum \xi_i^2 = \frac{D}{D^2} \sigma^2$

$$= \frac{\sigma^2}{D}$$

and $y = mx + C$

$$y_i = m(\xi_i + \bar{x}) + C \rightarrow \begin{cases} b \equiv m\bar{x} + C = \bar{y} = \frac{1}{N}\sum y_i \\ C = b - m\bar{x} \end{cases}$$

$$\sigma_b^2(\text{int}) = \frac{\sigma^2}{N}$$

$$\sigma_c^2(\text{int}) = \frac{\sigma^2}{N} - \frac{\bar{x}^2}{D}\sigma^2$$

$$= \left(\frac{1}{N} - \frac{\bar{x}^2}{D}\right)\sigma^2$$

Now we should compute σ^2 which is related to (y_i)

We saw that $\sigma^2 = \frac{N}{N-2}\langle s^2\rangle$

$$s^2 = \frac{1}{N}\sum d_i^2 \quad , \quad d_i \equiv y_i - \underbrace{(m\xi_i + b)}_{\text{optimum value}}$$

$$d_i = y_i - (m\xi_i + b) = \underbrace{e_i - Y_i}_{} - (m\xi_i + b)$$

$e_i \equiv y_i - Y_i \rightarrow$ Real value

$$d_i = e_i - \left[(m - M)\xi_i + (b - B)\right] \qquad \boxed{Y_i = M\xi_i + B}$$

Recall that

$$b = \frac{1}{N}\sum y_i \quad , \quad m = \frac{\sum \xi_i y_i}{D}$$

$$m - M = \frac{\sum \xi_i y_i}{D} - \frac{\sum \xi Y_i}{D} = \frac{1}{D}\sum \xi_i \underbrace{(y_i - Y_i)}_{e_i}$$

$$b - B = \frac{1}{N}\sum e_i$$

Plug in above terms in d_i and use $\sum \xi_i = 0$ we have

$$\sum_i d_i^2 = \underbrace{\sum e_i^2}_{=N\sigma^2} - \underbrace{\frac{1}{D}\left(\sum \xi_i e_i\right)^2}_{=\sigma^2} - \underbrace{\frac{1}{N}\left(\sum e_i\right)^2}_{=\sigma^2}$$

$$= N\sigma^2 - \underbrace{\sigma^2}_{\frac{1}{D}\left[\sigma^2\{\sum \xi_i^2 + 2\sum \xi_i \xi_j\}\right] = \sigma^2\frac{D}{D}} - \sigma^2$$

$$\sum d_i^2 = (N-2)\sigma^2 = N S^2 \Rightarrow \sigma^2 = \frac{N}{N-2}\langle S^2 \rangle$$

$$\text{So} \begin{cases} \sigma_m^2(\text{int}) = \dfrac{\sigma^2}{D} = \dfrac{1}{D}\dfrac{1}{N-2}\sum d_i^2 \\[2ex] \sigma_c^2(\text{int}) = \left(\dfrac{1}{N} + \dfrac{\bar{x}^2}{D}\right)\dfrac{\sum d_i^2}{N-2} \end{cases}$$

※ Exercise: Do above approach with take into account σ_x