به نام خدا

Multi-Fractal Analysis

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http://faculties.sbu.ac.ir/~movahed

Workshop on Time series analysis - Sharif University of Technology

24 Tir (1389)

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Outline

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- Motivations
- A brief explanation about Self-similar and self-affine models
- Novel methods in multifractal analysis
- Relation between so-called Hurst exponent and other scaling exponent in 1 and 2 dimensions
- Trends and undesired noise in time series
  1) Polynomial trends
  2) Sinusoidal trends
- Detrending procedures: F-DFA, SVD and EMD

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- Detrending procedures: F-DFA, SVD and EMD
- Summary

Some relevant references

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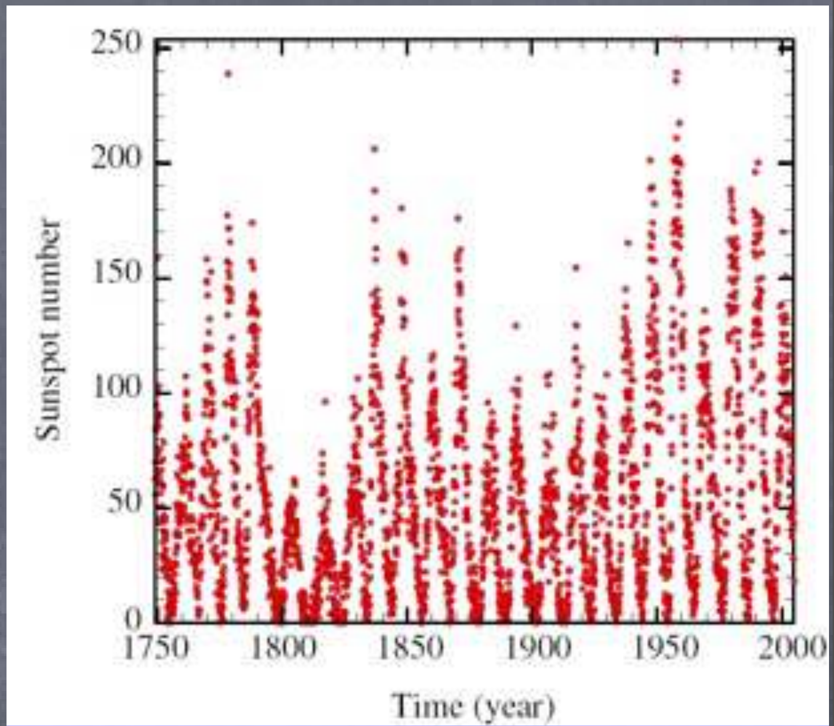
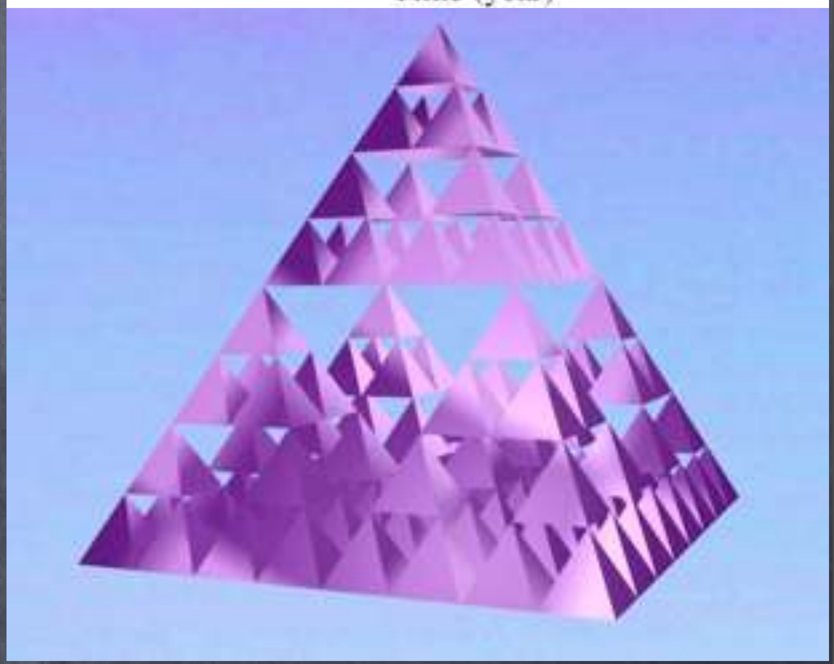
- http://sharif.ir/~rahimitabar/
- http://faculties.sbu.ac.it/~movahed
- arXiv:0804.0747
- PRE, 74, 061104 (2006)
- Physica A 316, 87-114 (2002)
- PRE 71, 011104 (2005); arXiv:physics/0103018
- Physica A 357, 447-454 (2005); Physica A 354, 182-198 (2005); Chaos, Solitons and fractals 26, 777-784 (2005)
- PNAS, 104, 38 (2007)
- Physiol. Meas. 23 (2002) R1-R38

Glossary

- Complex system: A system consisting of many non-linear components.
- Time series: One dimensional array representing value of an observable based on dynamical variable so-called time.
- Scaling law: A power law function describing the behavior of a typical physical quantity.
- Fractal and multifractal systems: A typical system which characterized by a scaling law with non-integer exponent in all scaling ranges. On the other hand, multifractal has infinite number of different fractal exponents. Each of them are valid in proper scaling range.



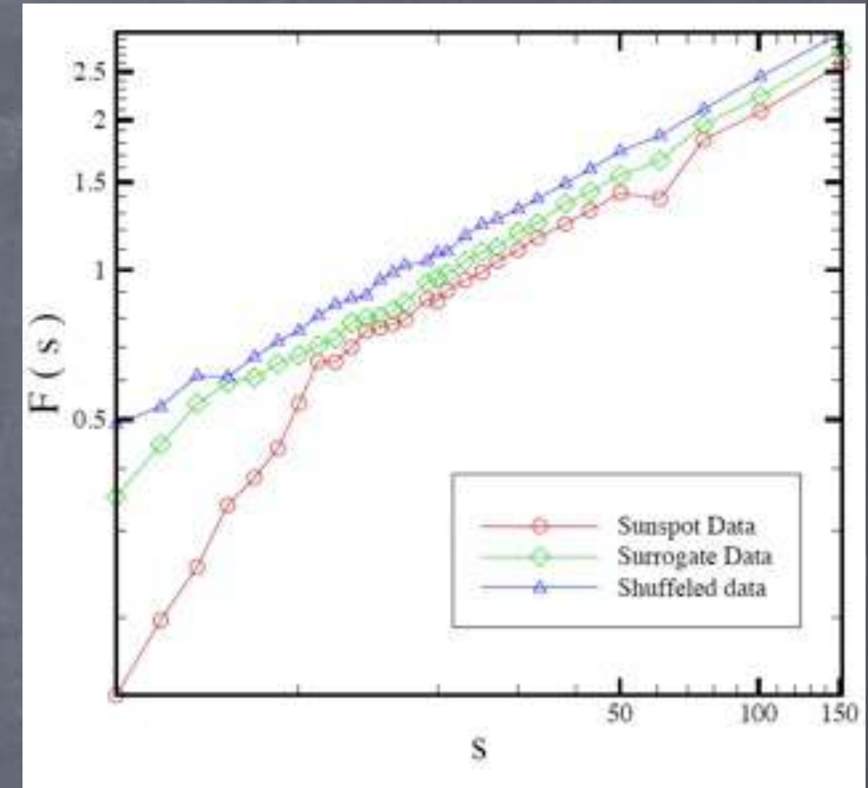


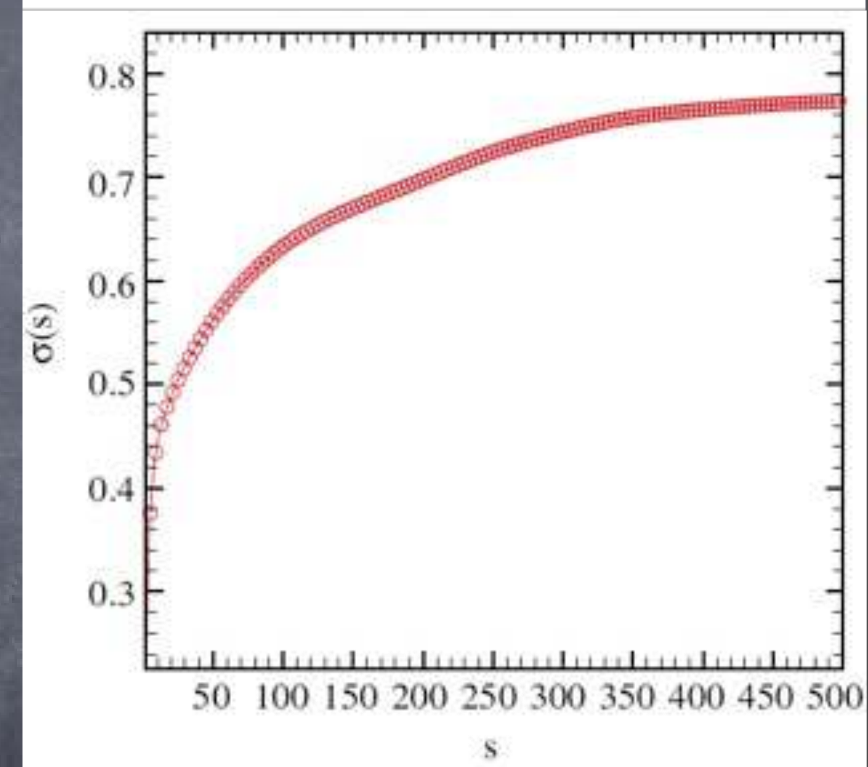
- Self-similar and self-affine: Magnification of system's parts in every directions have same scaling exponent for matching to whole of system. While self-affinity is a generalization for anisotropic scaling behavior.

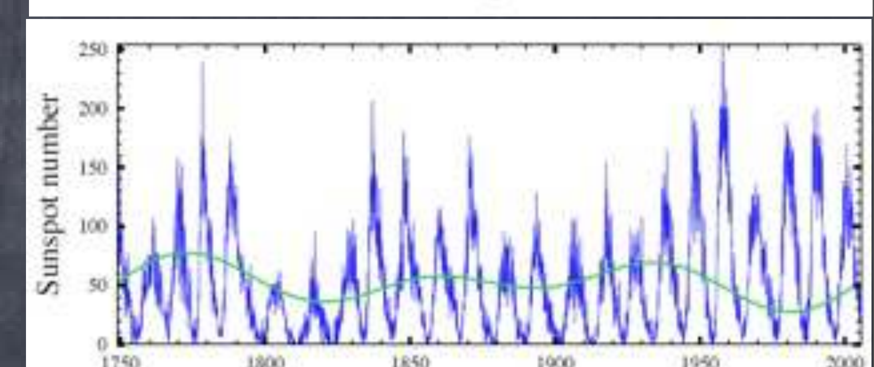
- Cross-over: Changing in the scaling behavior

- Non-stationary: The weak definition is concerned to changing the mean standard deviation of time series with time. Strong definition of stationarity requires that all moments remain constant. Usually external affects cause nonstationarity in time series.

- Trend and detrending: It is an intrinsically fitted monotonic function or a function in which there can be at most one extremum within a given data span. Detrending is the operation of removing trend







A brief History on Complex system science

A brief History on Complex system science

- ~1700 A.D. Gottfried leibniz
- ~1872 A.D. Karl Weierstrass
- ~1904 A.D. Helge von Koch
- ~1915 A.D. Waclaw Seirpinski
- ~1951 A.D. H. E. Hurst
- ~1968 A.D. B. B. Mandelbroat

Model construction

- To investigate the evolution of phenomena in the nature and probably track their future situations.
- It should be a simple from mathematical point of view.

Natural time series

- Geophysics time series: temperature, precipitation, water runoff, seismic events, climate dynamics and so on.
- Medical and physiological time series: Heartbeat, blood pressure, glucose level, gene expression data and so on.
- Astrophysical time series: X-ray and cosmic ray, sunspot, CMB (actually not time series),....
- Social and technical time series: Traffic, internet, Finance, language characteristics, chemistry a petroleum, ...
- Physics data: surface roughness, spectroscopy, ...

Why Fractal and multifractal Analysis?

- Prediction of the future behavior of the systems
- Classification of various systems from complex systems point of view
- Find the universality properties of underlying systems

Multifractality in human heartbeat dynamics

Plamen Ch. Ivanov*†, Luís A. Nunes Amaral
Ary L. Goldberger†, Shlomo Havlin‡,
Michael G. Rosenblum§, Zbigniew R. Struzik

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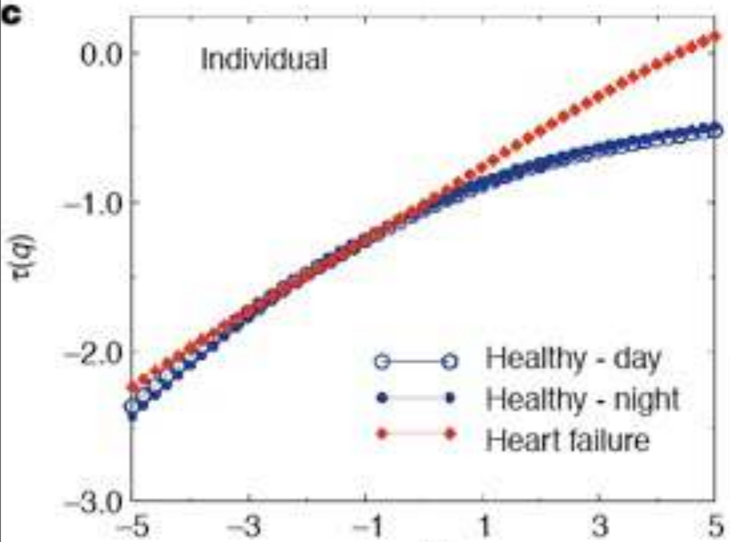

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Physica A 380 (2007) 250–258
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Long range correlations in the heart rate variability following the injury of cardiac arrest

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Scaling and correlation in financial time series

P. Gopikrishnan[a,*], V. Plerou[a], Y. Liu[a], L.A.N. Amaral[a],
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Statistical physics and physiology: Monofractal and multifractal approaches

H.E. Stanley[a,*], L.A.N. Amaral[a,b], A.L. Goldberger[b], S. Havlin[a,c],
P.Ch. Ivanov[a,b], C.-K. Peng[b]

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Multifractal detrended fluctuation analysis of pressure fluctuation signals in an impinging entrained-flow gasifier

Miaoren Niu*, Fuchen Wang, Qinfeng Liang, Guangsuo Yu, Zunhong Yu

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Physica A 380 (2007) 418–428
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Music walk, fractal geometry in music

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Statistical physics and physiology: Monofractal and multifractal approaches

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Problems and Discrepancies regarding to Observations and Models

- Direct computation and determination

- Indirect computation and determination

Problems and Discrepancies regarding to Observations and Models

- ~~Direct computation and determination~~

Trend and unknown noise

- Indirect computation and determination

Self-affinity in time series

- Suppose a time series as:

$$y:\{y(i)\} \quad i=1,\ldots,N$$

$$i \rightarrow a \times i$$

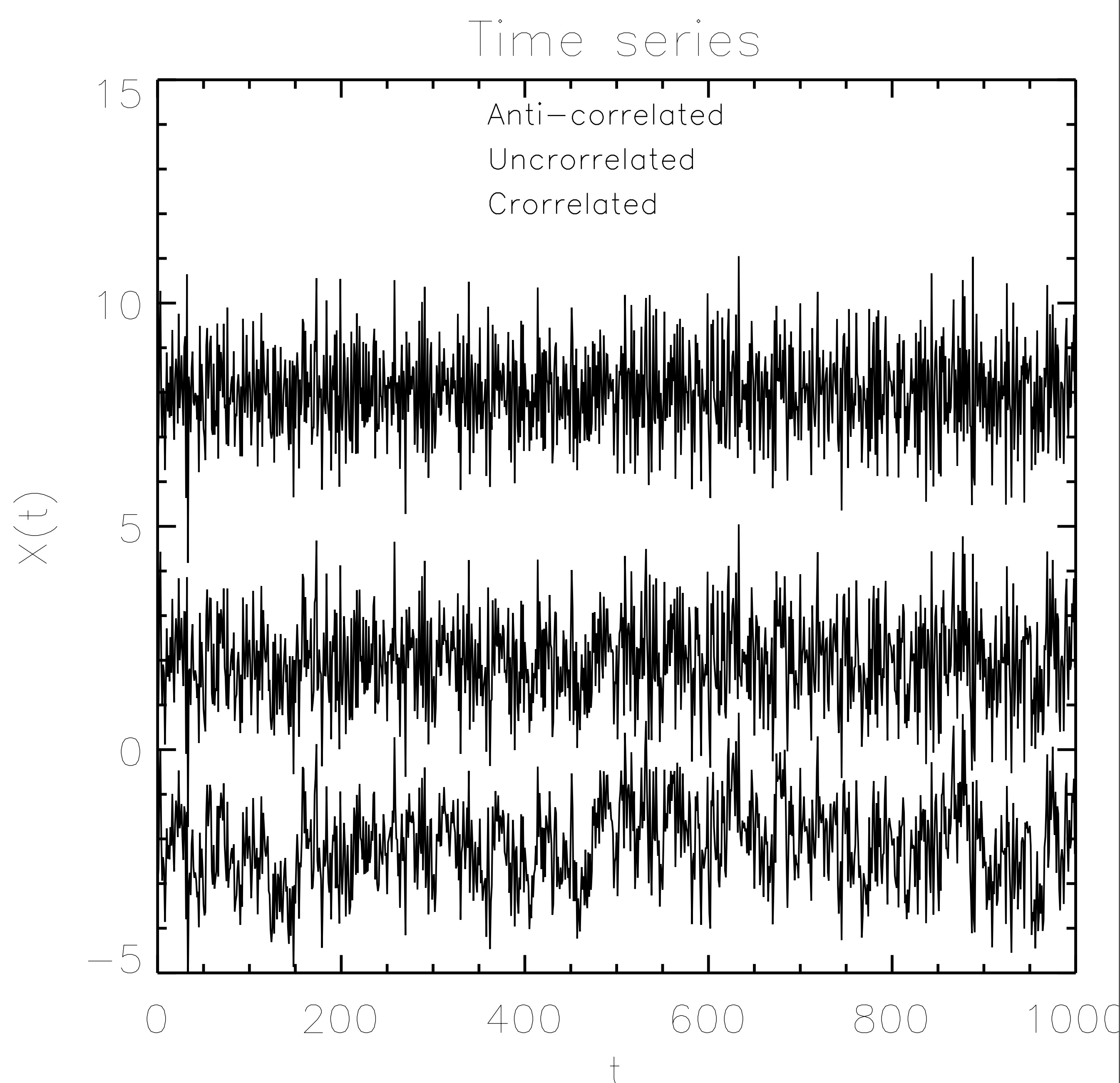
So-called Hurst exponent

$$y(a \times i) = a^{H} y(i)$$

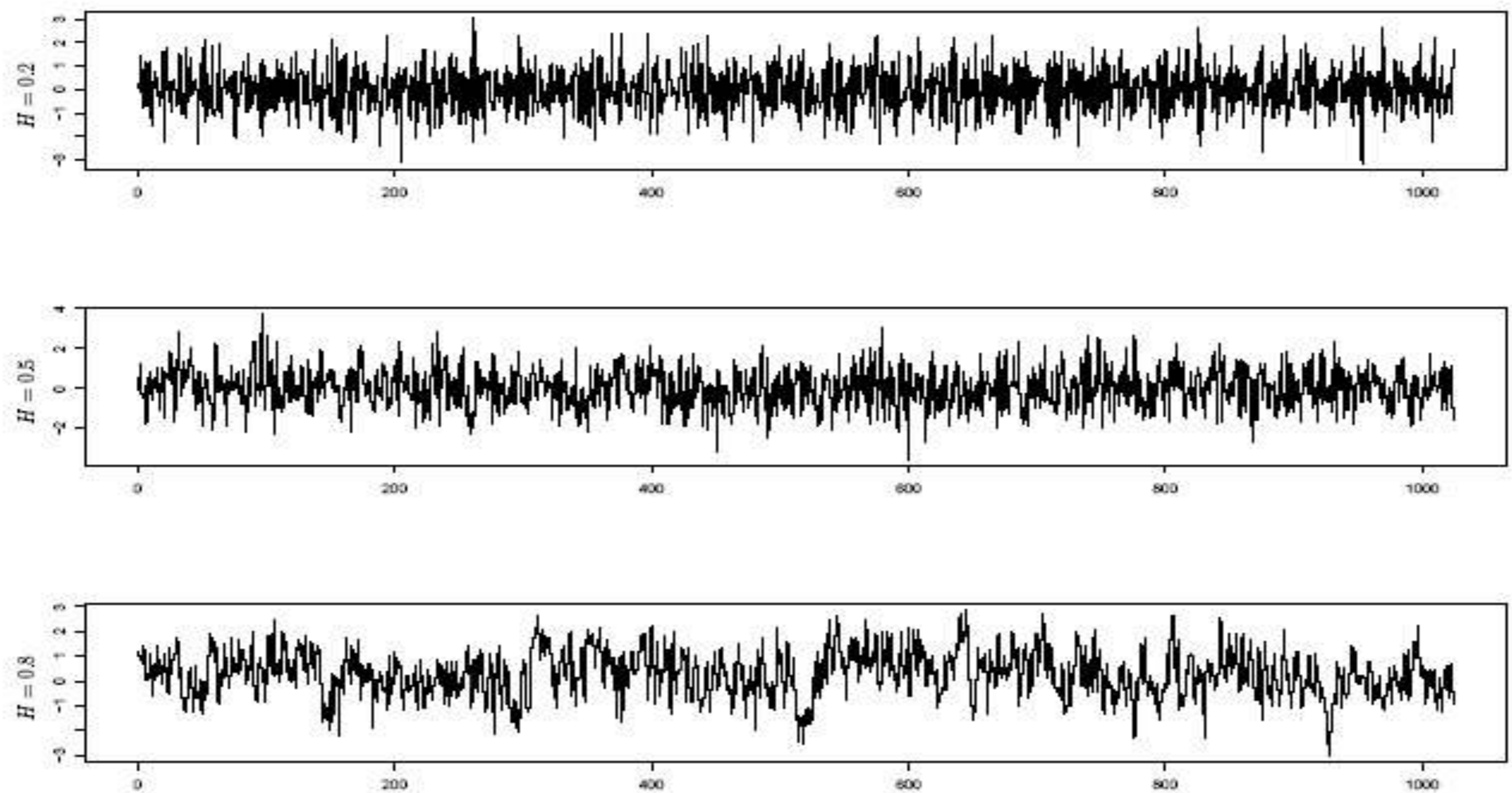
$$y(i) = x(1) + x(2) + x(3) + \ldots + x(i) = i^{H} x(1)$$

Classification of time series based on Hurst exponent

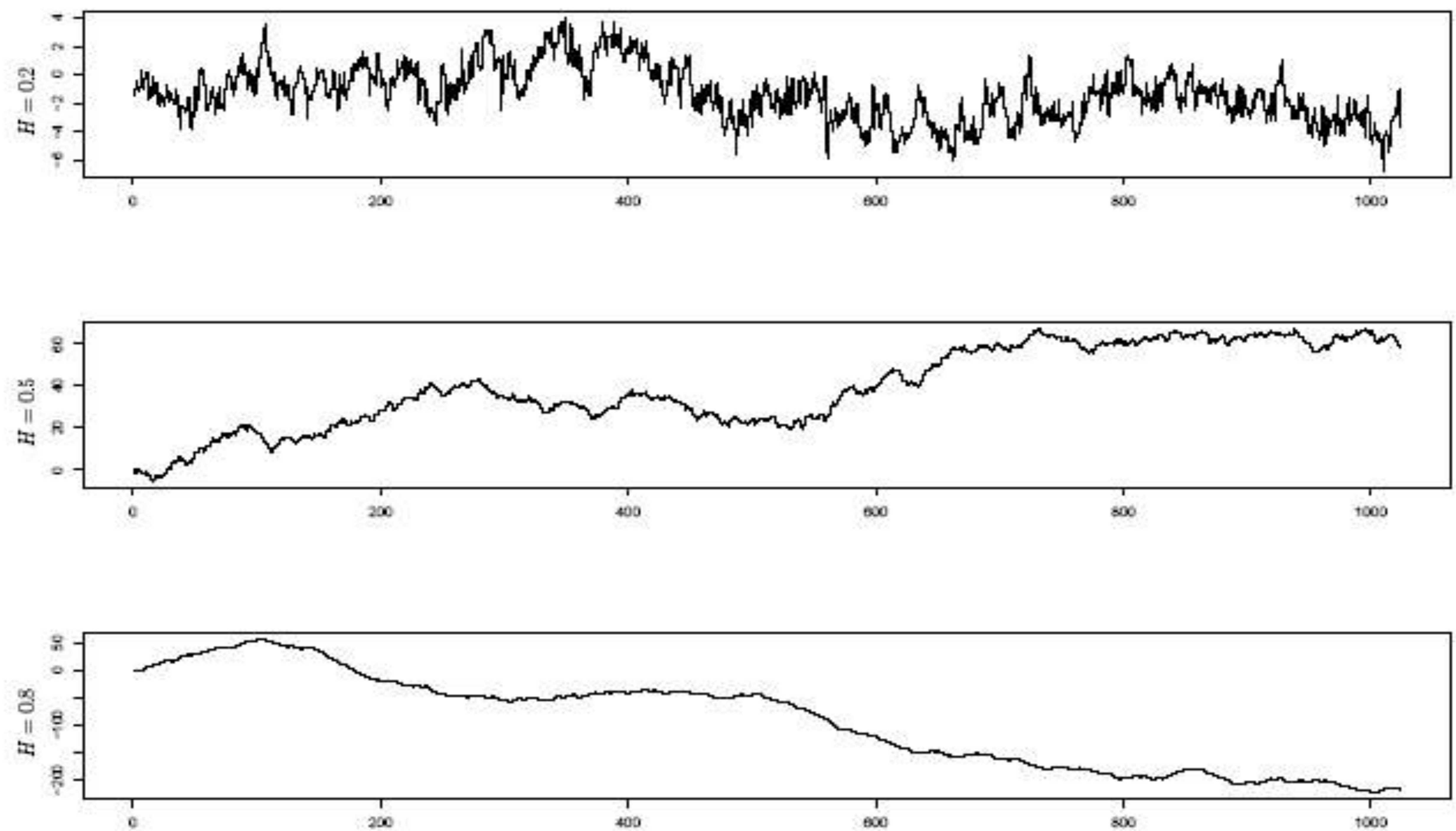
- Anti-correlated : H<0.5
- Uncorrelated: H=0.5
- Correlated: H>0.5



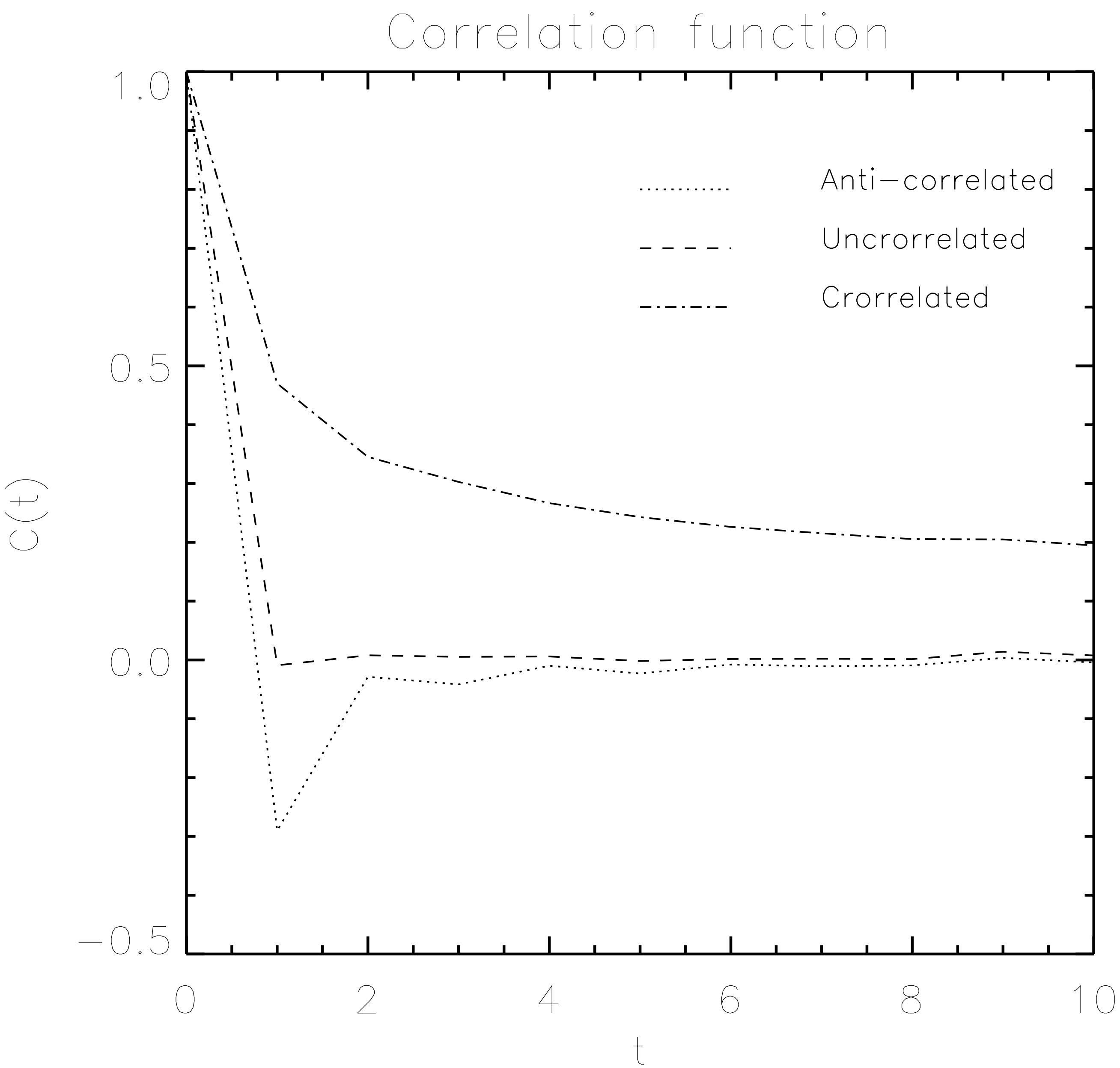
Fractional Gaussian Noise

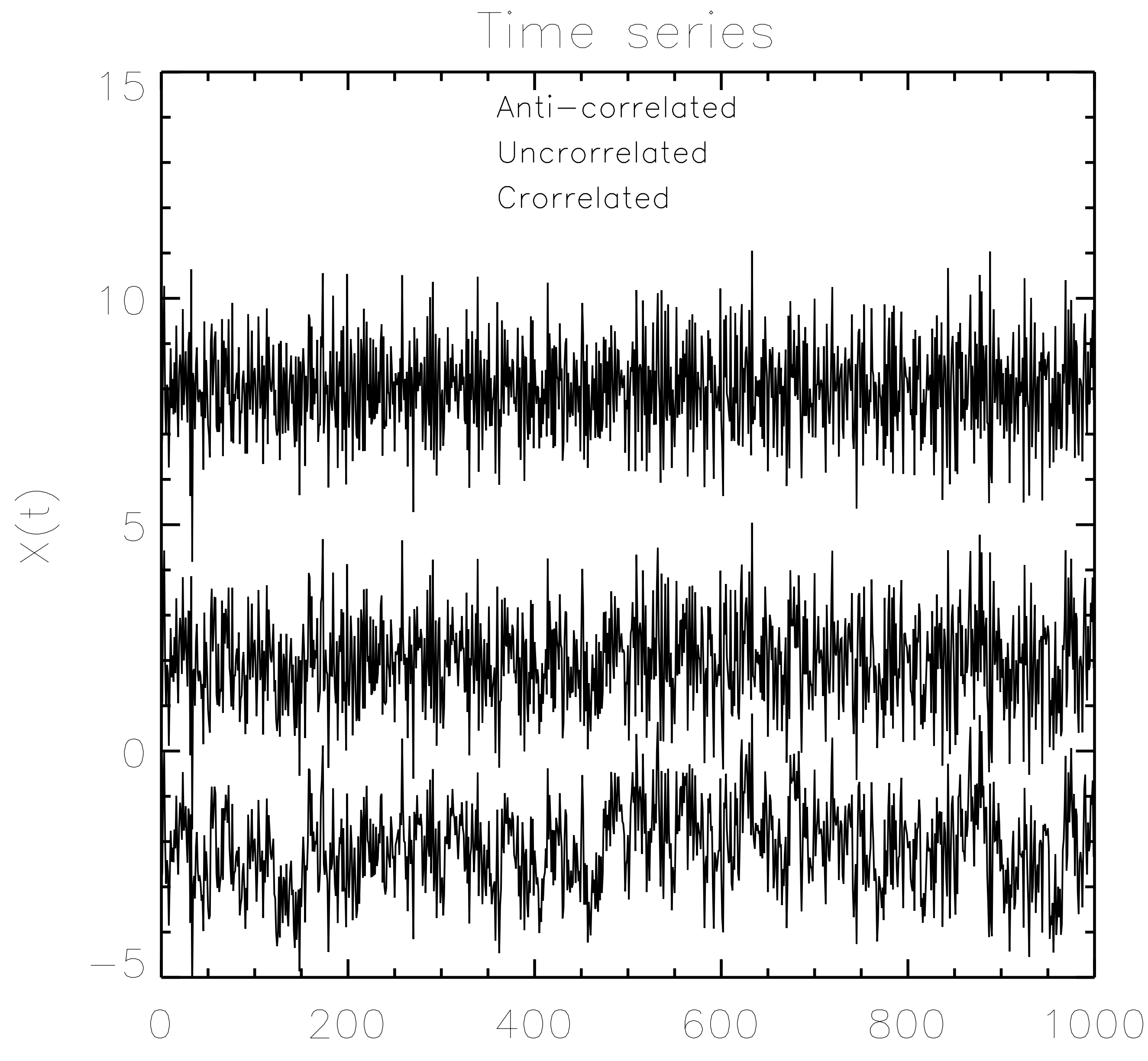


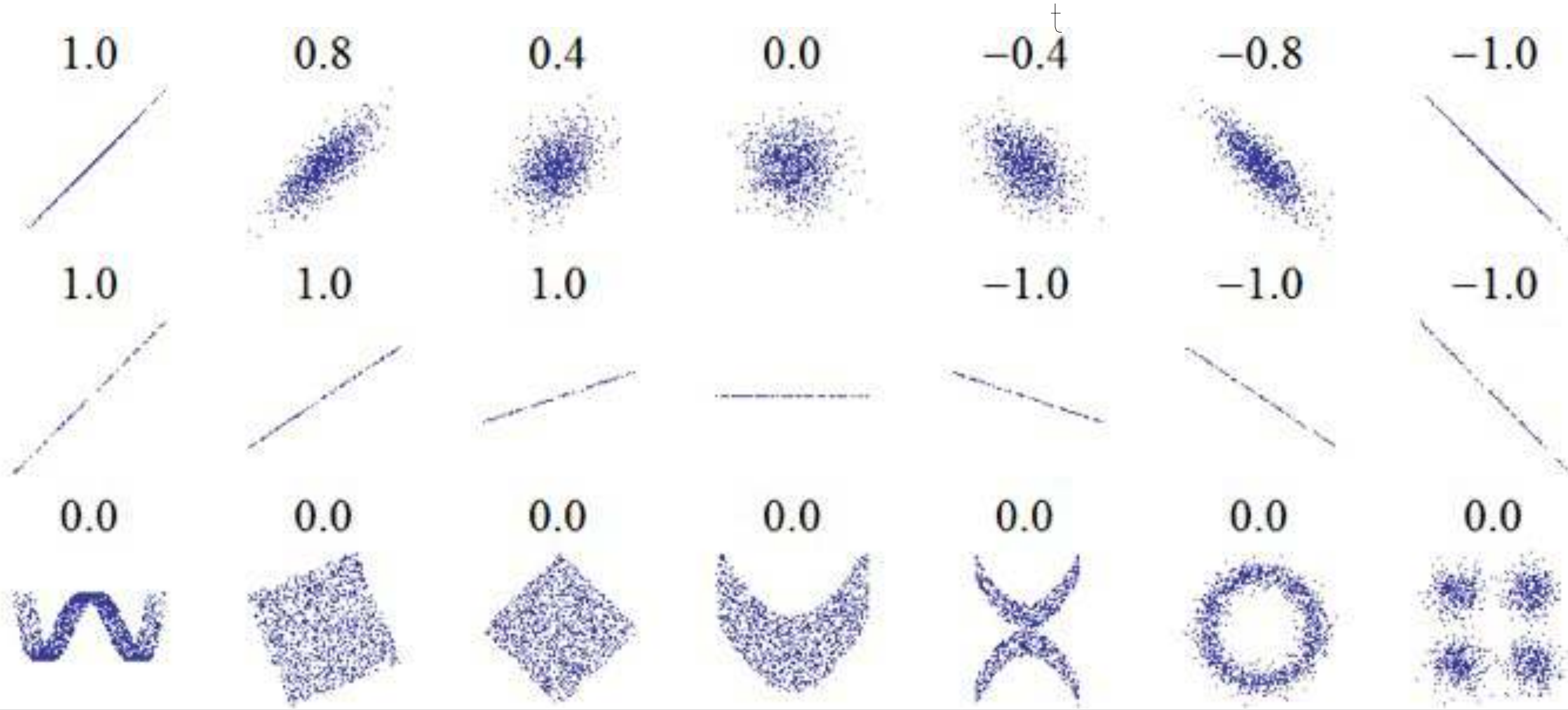
Fractional Brownian Motion



Thursday, July 15, 2010







S. Kimiagar, M. Sadegh Movahed et. al., arXiv:0710.5270

Novel Fractal Analysis methods

- Hurst' rescaled range (R/S) analysis : By Hurst (1951)
- Scaled windowed variance analysis (SWA) : By Mandelbort (1985)
- Dispersional analysis (Disp) : By Bassingthwaighte (1988)
- Detrended fluctuation analysis (DFA) : By Peng (1994)
- Some state-of-the-art algorithm based on previous idea such as: MF-DFA, MF-DCCA, MF-TWDFA, DMA (BDMA & CDMA), WTMM

Detrending methods

- Parametric: Done in DFA
- Non-parametric: Empirical mode decomposition (EMD)

I must point out that now a days there are some challenge regarding to Detrending methods in multifractal analyses

Description and Application of mentioned methods

- Part A: For stationary case without trends
- Part B: For non-stationary case with trends

SWV method

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} \left[x_k - \langle x \rangle \right], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}\,(N/s)$ non-overlapping segments of equal lengths s.

$$SWV(s) = \left(\frac{١}{s} \sum_{i=١}^{s} [Y(i) - \langle Y(s) \rangle]^{٢} \right)^{١/٢}$$

$$SWV(s) \sim s^{H}$$

R/S method

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} [x_k - \langle x \rangle], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}\,(N/s)$ non-overlapping segments of equal lengths s.

$$\begin{aligned} R(s) &= Max\{Y(s)\} - Min\{Y(s)\} \\ S(s) &= \left(\frac{١}{s} \sum_{i=١}^{s} [x(i) - \langle X \rangle]^{٢} \right)^{١/٢}, \;\; s = ١, \ldots, N \end{aligned}$$

$$R(s)/S \sim s^{H}$$

Dispersional method

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} [x_k - \langle x \rangle], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}(N/s)$ non-overlapping segments of equal lengths s.

$$\mu(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} Y[(\nu - 1)s + i]$$

$$\langle \mu(s) \rangle = \frac{1}{N_s} \sum_{\nu=1}^{N_s} \mu(\nu, s)$$

$$M(s) = \frac{1}{N_s} \sum_{\nu=1}^{N_s} [\mu(\nu, s) - \langle \mu(s) \rangle]$$

$$M(s) \sim s^{2H}$$

Multi-Fractal Detrended Fluctuation in 1D

DFAm remove trend of order m in profile or trend of order m-1 in original seris

- *Step 1.* Determine the 'profile'

$$Y(i) \equiv \sum_{k=1}^{i} [x_k - \langle x \rangle], \qquad i = 1, \ldots, N.$$

- *Step 2.* Divide the profile $Y(i)$ into $N_s \equiv \mathrm{int}(N/s)$ non-overlapping segments of equal lengths s.

- *Step 3.* Calculate the local trend for each of the $2N_s$ segments by a least squares fit of the series. Then determine the variance

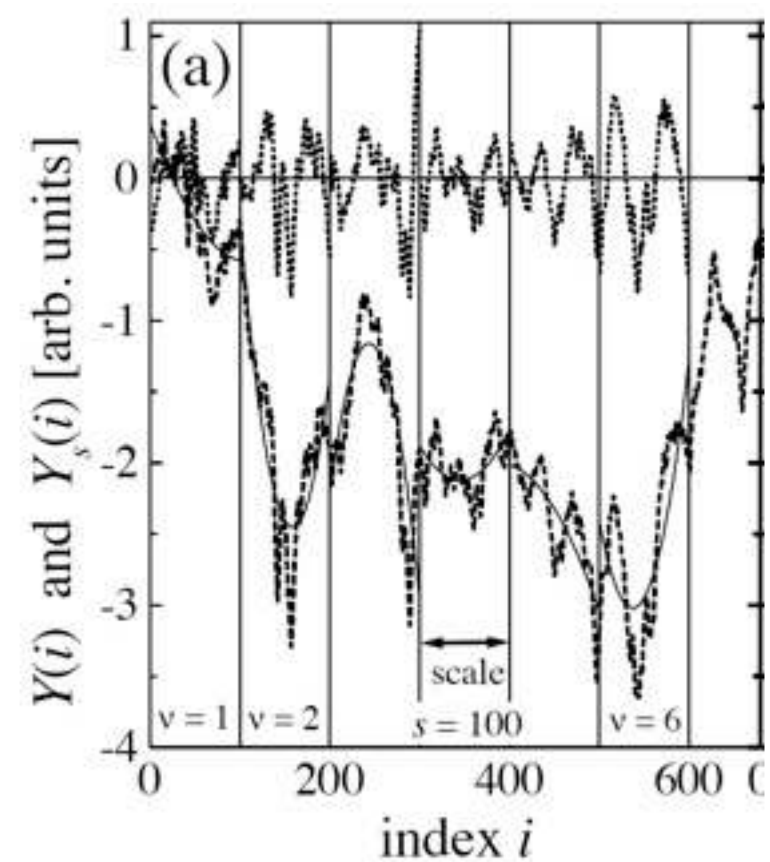
$$F^2(s,\nu) \equiv \frac{1}{s} \sum_{i=1}^{s} \{Y[(\nu-1)s+i] - y_\nu(i)\}^2$$

for each segment ν, $\nu = 1, \ldots, N_s$, and

$$F^2(s,\nu) \equiv \frac{1}{s} \sum_{i=1}^{s} \{Y[N-(\nu-N_s)s+i] - y_\nu(i)\}^2,$$

for $\nu = N_s+1, \ldots, 2N_s$.

Here, $y_\nu(i)$ is the fitting polynomial in segment ν. Stop



- *Step 4*. Average over all segments to obtain the qth-order fluctuation function, defined as

$$F_q(s) \equiv \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} \left[F^2(s,\nu)\right]^{q/2} \right\}^{1/q}$$

$F_q(s)$ is only defined for $s \geq m+2$.

- *Step 5*. Determine the scaling behaviour of the fluctuation functions by analysing log–log plots of $F_q(s)$ versus s for each value of q. If the series x_i are long range power law correlated, $F_q(s)$ increases, for large values of s, as a power law,

$$F_q(s) \sim s^{h(q)}.$$

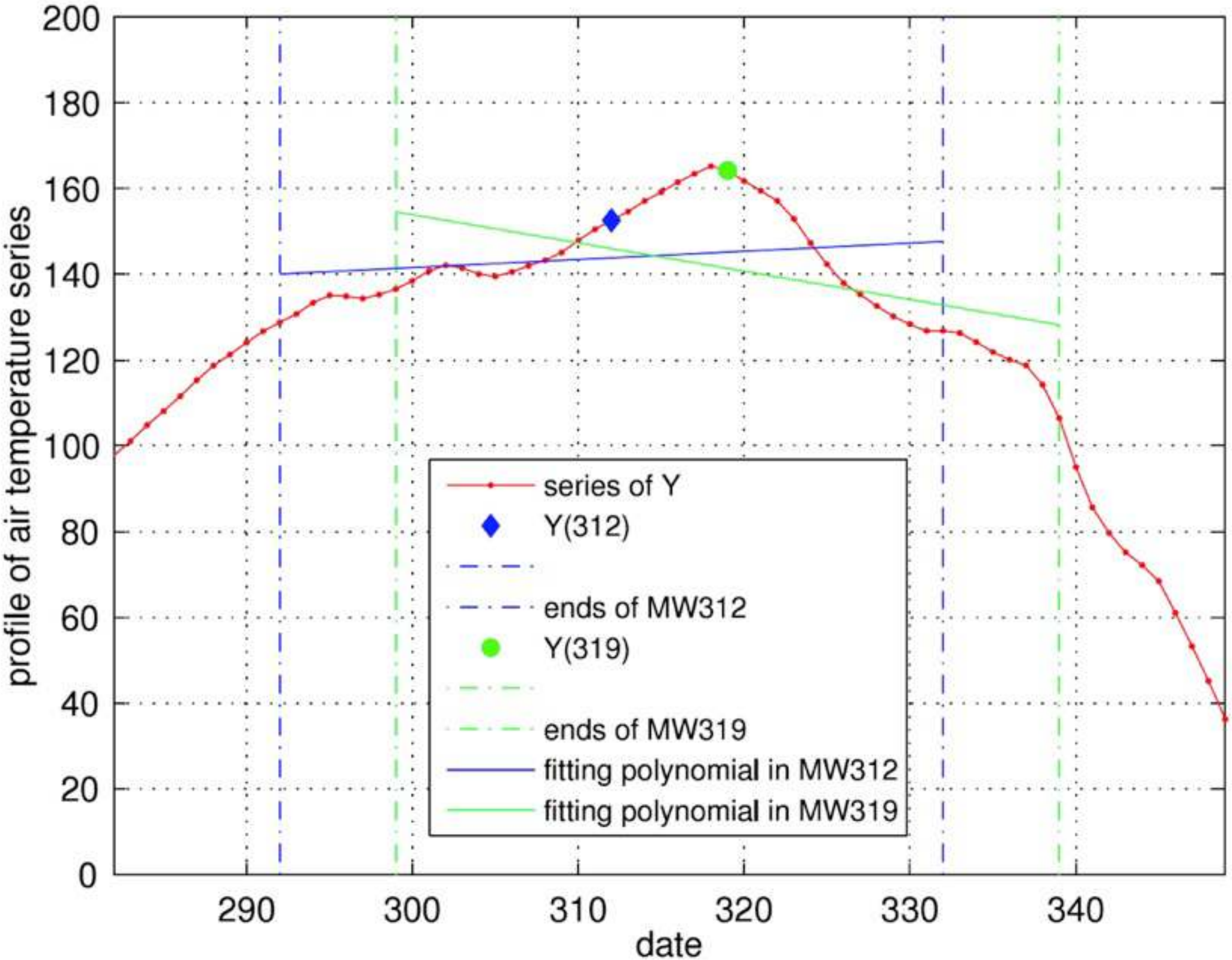
$\{X\} : \{F_q(s)\}$ $\qquad$ $\{\Theta\} : \{h(q)\}$

$$P(h(q)|X) = \frac{\mathcal{L}(X|h(q))P(h(q))}{\int \mathcal{L}(X|h(q))dh(q)} \qquad \mathcal{L}(X|h(q)) \sim \exp\left(\frac{-\chi^2(h(q))}{2}\right)$$

$$\chi^2(h(q)) = \int ds \frac{[F_{\text{obs.}}(s) - F_{\text{The.}}(s; h(q))]^2}{\sigma^2_{\text{obs.}}(s)}$$

$$68.3\% = \int_{-\sigma^-}^{+\sigma^+} \mathcal{L}(X|h(q))dh(q) \qquad h^{+\sigma^+}_{-\sigma^-}$$

Jan W. Kantelhart, et. al., arXiv:physics/0202070; M. Sadegh Movahed et. al., arXiv:physics/0508149
S. Hajian and M. Sadegh Movahed, arXiv:0908.0132



Yu Zhou and Yee Leung, JSTAT P06021 (2010)

h(2) and Hurst exponent in DFA1 for fGn

$$F^2(s) \equiv \frac{1}{N_s}\sum_{\nu=1}^{N_s}[F^2(s;\nu)], \qquad (1$$

$$= \left\langle [F^2(s;\nu)] \right\rangle_\nu ,$$

$$\equiv \mathcal{C}_H s^{2H},$$

$$F^2(s;\nu) = \frac{1}{s}\sum_{i=1}^{s}[Y_\nu(i) - y_\nu(i)]^2 \qquad (2$$

$$Y(i) = \sum_{k=1}^{i} x(k) - \langle x \rangle \qquad (3$$

$$y_\nu(i) = a_\nu + b_\nu i \qquad (4$$

$$\left\langle [F^2(s;\nu)] \right\rangle = \left\langle \frac{1}{s} \sum_{i=1}^{s} [Y(i) - a - bi]^2 \right\rangle$$

$$\simeq \left\langle \frac{1}{s} \sum_{i=1}^{s} Y(i)^2 \right\rangle + \langle a^2 \rangle + \frac{s^2}{3} \langle b^2 \rangle + s \langle ab \rangle - 2 \left\langle \frac{a}{s} \sum_{i=1}^{s} Y(i) \right\rangle - 2 \left\langle \frac{b}{s} \sum_{i=1}^{s} iY(i) \right\rangle$$

$$Y(i) = i^H x \longrightarrow Y(i) - Y(k) = |i-k|^H x$$

$$\langle [Y(i) - Y(k)]^2 \rangle = \sigma^2 |i-k|^{2H} \qquad \sigma^2 = \langle x(i)^2 \rangle \qquad \langle Y(i)^2 \rangle = \sigma^2 i^{2H}$$

$$\langle Y(i) Y(k) \rangle = \frac{\sigma^2}{2} [i^{2H} + k^{2H} - |i-k|^{2H}]$$

$$\left\langle [F^2(s;\nu)] \right\rangle_\nu = \mathcal{C}_{\mathcal{H}} (s)^{2H}$$

$$\sum_{i,j=1}^{s} \langle iY(i)Y(j) \rangle = \frac{\sigma^2}{2} \sum_{i,j=1}^{s} \left(i^{2H+1} + ij^{2H} - i|i-j|^{2H} \right),$$

$$= \frac{\sigma^2}{2} \sum_{i,j=1}^{s} \left(i^{2H+1} + ij^{2H} \right) - \frac{\sigma^2}{2} \sum_{i=1}^{s} \sum_{j=1}^{i} i(i-j)^{2H} - \frac{\sigma^2}{2} \sum_{i=1}^{s} \sum_{j=i}^{s} i(j-i)^{2H},$$

$$\sim \frac{\sigma^2}{2} \left(\frac{s^{2H+3}}{2H+2} + \frac{s^{2H+3}}{2(2H+1)} \right) - \frac{\sigma^2}{2} \sum_{i=1}^{s} i^{2H+2} \left(\int_0^1 (1-x)^{2H} \mathrm{d}x - \int_0^1 x(1-x)^{2H} \mathrm{d}x \right)$$

$$= \frac{\sigma^2 s^{2H+3}}{4} \left(\frac{2}{H+1} - \frac{1}{2H+1} \right)$$

$$
\begin{aligned}
\sum_{i,j=1}^{s} \langle Y(i)Y(j)\rangle &= \frac{\sigma^2}{2} \sum_{i,j=1}^{s} \left(i^{2H} + j^{2H} - |i-j|^{2H}\right), \\
&= \frac{\sigma^2}{2} \sum_{i,j=1}^{s} \left(i^{2H} + j^{2H}\right) - \sigma^2 \sum_{i=1}^{s}\sum_{j=1}^{i} (i-j)^{2H}, \\
&\sim \sigma^2 \left(\frac{s^{2H+2}}{2H+1} - \sum_{i=1}^{s} i^{2H+1} \int_0^1 (1-x)^{2H}\right), \\
&\sim \sigma^2 s^{2H+2} \left(\frac{1}{2H+1} - \frac{1}{(2H+2)(2H+1)}\right).
\end{aligned}
$$

$$
\begin{aligned}
F^2(s) &\equiv \frac{1}{N_s} \sum_{\nu=1}^{N_s} [F^2(s;\nu)], \\
&= \left\langle [F^2(s;\nu)] \right\rangle_{\nu}, \\
&\equiv \mathcal{C}_H s^{2H},
\end{aligned}
$$

$$
\mathcal{C}_{\mathcal{H}} = \frac{\sigma^2}{(2H+1)} - \frac{4\sigma^2}{2H+2} + 3\sigma^2 \left(\frac{2}{H+1} - \frac{1}{2H+1}\right) - \frac{3\sigma^2}{(H+1)} \left(1 - \frac{1}{(H+1)(2H+1)}\right)
$$

$$
h(q=2) = H
$$

M. S. Taqqu et. al., Fractals, Vol. 3, No. 4 (1995)M. Sadegh Movahed et. al., arXiv:physics/0608056

h(2) and Hurst exponent in DFA1 for fBm

$$
\begin{aligned}
\langle [F^2(s,\nu)] \rangle &= \left\langle \frac{1}{s}\sum_{i=1}^{s}(Y(i)-a-bi)^2 \right\rangle \\
&\simeq \left\langle \frac{1}{s}\sum_{i=1}^{s}Y(i)^2 \right\rangle + \langle a^2 \rangle + \frac{s^2}{3}\langle b^2 \rangle \\
&- 2\left\langle \frac{a}{s}\sum_{i=1}^{s}y(i) \right\rangle - 2\left\langle \frac{b}{s}\sum_{i=1}^{s}iY(i) \right\rangle + s\langle ab \rangle , \\
&= \left\langle \frac{1}{s}\sum_{i=1}^{s}Y(i)^2 \right\rangle - \frac{4}{s^2}\left\langle \left[\sum_{i=1}^{s}Y(i)\right]^2 \right\rangle \\
&- \frac{12}{s^4}\left\langle \left[\sum_{i=1}^{s}iY(i)\right]^2 \right\rangle + \frac{12}{s^3}\left\langle \sum_{i=1}^{s}iY(i)\sum_{i=1}^{s}Y(i) \right\rangle \\
&= \frac{A}{s} - \frac{4}{s^2}B - \frac{12}{s^4}D + \frac{12}{s^3}C
\end{aligned}
$$

$$
\begin{aligned}
x(i) &= Y(i) - Y(i-1) \\
u(i) &= x(i) - x(i-1),
\end{aligned}
$$

$$
\langle x(i)x(j) \rangle = \frac{\sigma^2}{2}\left[i^{2H} + j^{2H} - |i-j|^{2H}\right] ,
$$

$$
\langle Y(i)Y(j) \rangle = \frac{\sigma^2}{(H+1)^2}(ij)^{H+1} ,
$$

For fBm series

$$\left\langle \left[F^2(s,\nu)\right]\right\rangle_\nu = \mathcal{C}_H s^{2(H+1)},$$

$$\mathcal{C}_H = \frac{\sigma^2}{(2H+3)(H+1)^2} - \frac{4\sigma^2}{[(H+1)(H+2)]^2} - \frac{12\sigma^2}{[(H+1)(H+3)]^2} + \frac{12\sigma^2}{(H+1)^2(H+2)(H+3)}.$$

For fGn series

$$\left\langle [F^2(s;\nu)] \right\rangle_\nu = \mathcal{C}_{\mathcal{H}}(s)^{2H}$$

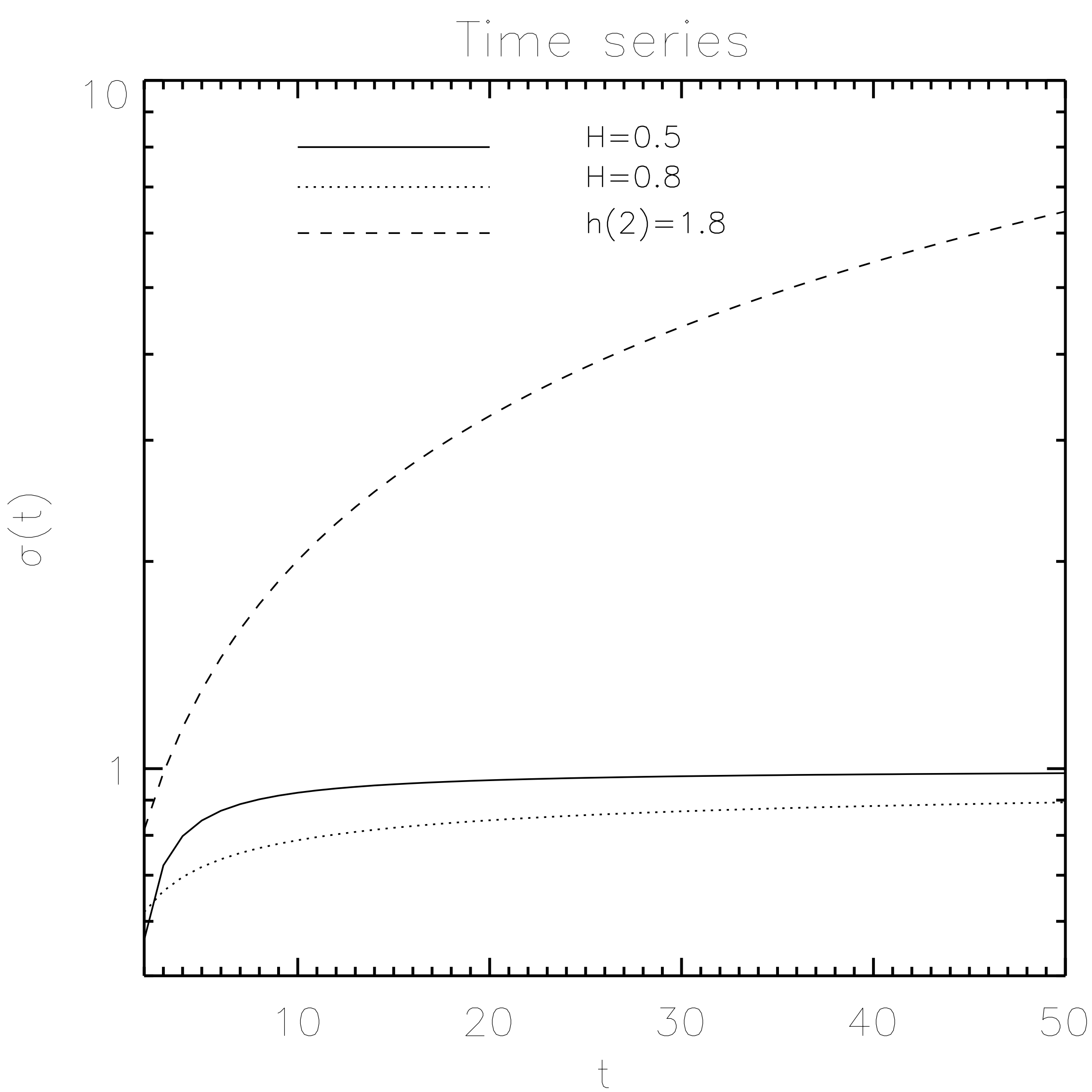
$$\mathcal{C}_{\mathcal{H}} = \frac{\sigma^2}{(2H+1)} - \frac{4\sigma^2}{2H+2} + 3\sigma^2\left(\frac{2}{H+1} - \frac{1}{2H+1}\right) - \frac{3\sigma^2}{(H+1)}\left(1 - \frac{1}{(H+1)(2H+1)}\right)$$

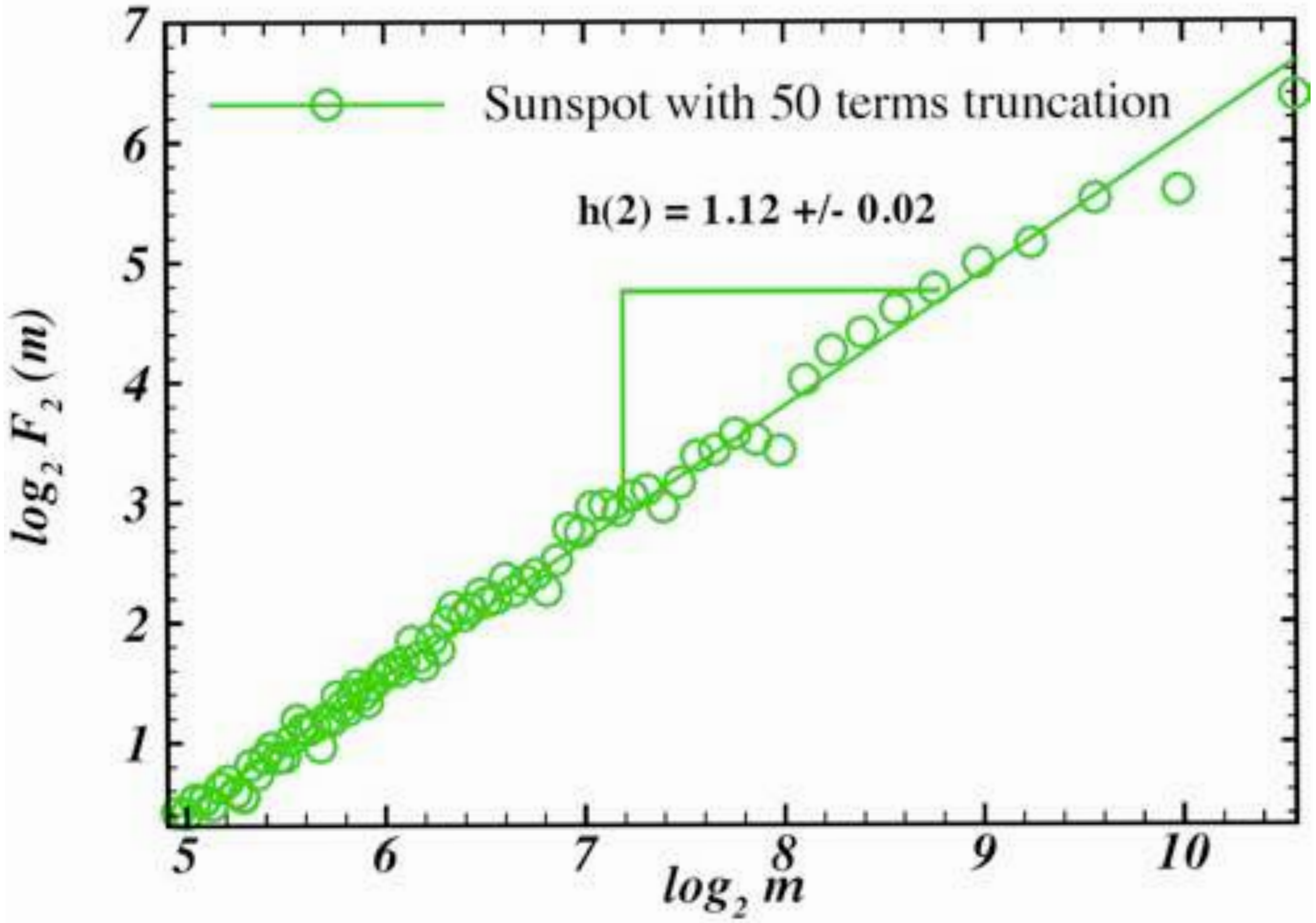
$$h(q=2) = H$$

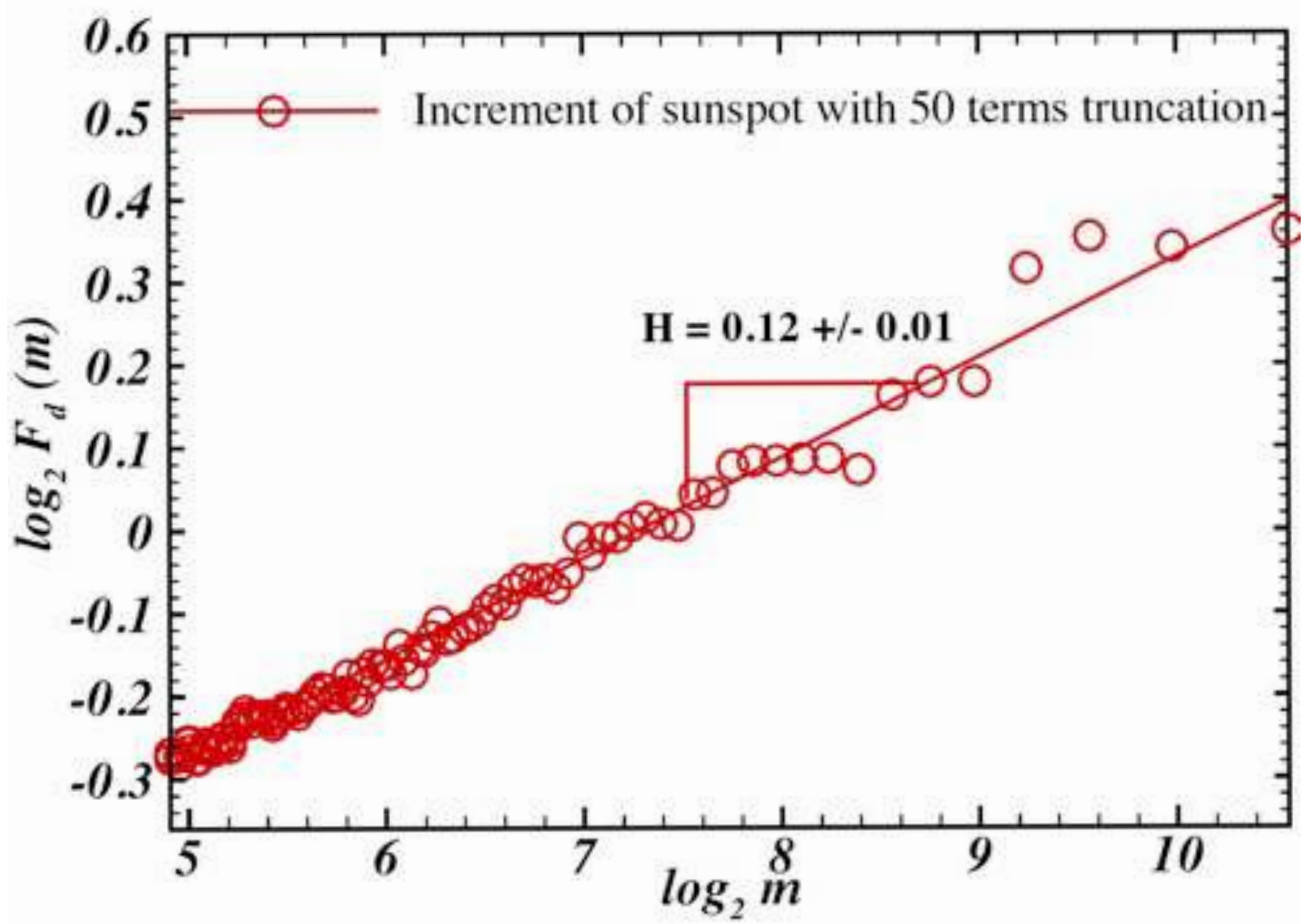
M. Sadegh Movahed et. al., arXiv:physics/0508149

For fBm: $h(q=2)>1$

For fGn: $h(q=2)<1$







Scaling exponents

- Multifractal scaling exponent $\tau(q) = qh(q) - 1$
- Generalized multifractal dimension $D(q) = \frac{\tau(q)}{q-1}$
- Autocorrelation exponent $\begin{cases} C(s) \sim s^{-\gamma} \\ C(i,j) \sim i^{-\gamma} + j^{-\gamma} - \lvert i-j \rvert^{-\gamma} \end{cases}$
- Power spectrum scaling exponent $S(\omega) \sim \omega^{-\beta}$
- Holder exponent $\alpha = \tau'(q)$

  $\alpha = h(q) + qh'(q)$
- Singularity spectrum $f(\alpha) = q[\alpha - h(q)] + 1$

Correlation and Hurst exponents

$$C(s) = \frac{\langle x(i+\tau)x(i)\rangle}{\sigma^2} \sim \tau^{-\gamma}$$

$$Y(s) = \sum_{k=1}^{s} x(k) = x(1)\times s^H$$

$$\langle Y(s)^2\rangle = \sigma^2 \times s^{2H}$$

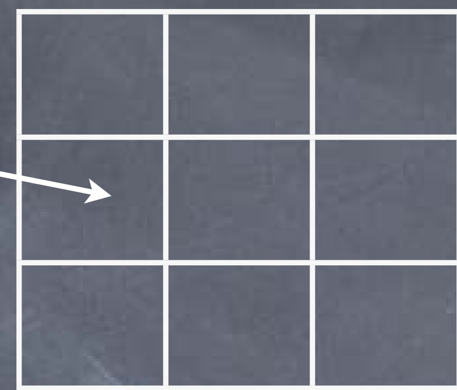
$$= \left\langle \left(\sum_{k=1}^{s} x(k)\right)^2 \right\rangle = \left\langle \sum_{k=1}^{s} x(k)^2 \right\rangle + \left\langle \sum_{k\neq j}^{s} x(k)x(j) \right\rangle$$

$$= i\sigma^2 + 2\sum_{j=1}^{s-1}(s-j)C(j) \sim s^{2-\gamma} = s^{2H} \rightarrow \gamma = 2-2H$$

$$for \qquad 0.5 < H < 1$$

Generalized fractal dimension based on partition function

$p(v,s)$



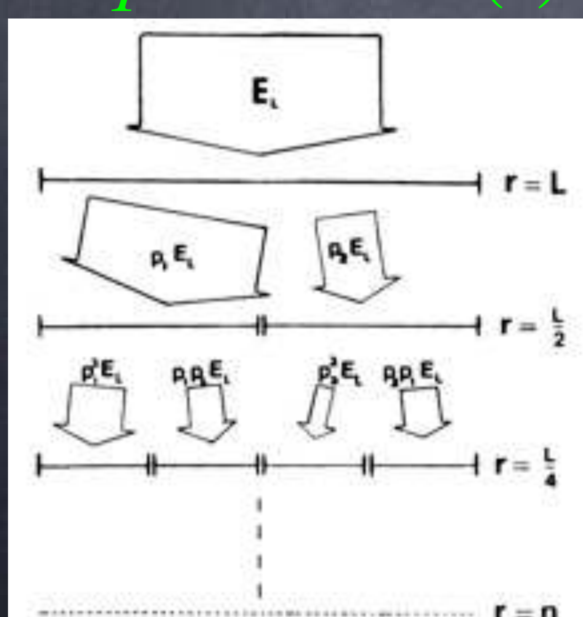
$$\sum_{v=1}^{N_s} p(v,s) = 1$$

$$Z_q(s) \equiv \sum_{v=1}^{N_s} |p(v,s)|^q \sim s^{\tau(q)}$$

$$D(q) \equiv \frac{1}{q-1} \lim_{s\to 0} \frac{\ln Z_q(s)}{\ln s} = \frac{\tau(q)}{1-q}$$

$$for \quad q=0 \qquad D(0) = D_f$$

$$for \quad q=1 \qquad D(1) \sim \sum p \ln p$$



$$p^2(v,s) = \frac{1}{s}\sum_{i=1}^{s}\{Y[(v-1)s+i] - y_v(i)\}^2$$

$$F_q(s) = \left(\frac{1}{N_s}\sum_{v=1}^{N_s}|p(v,s)|^q\right)^{1/q}$$

$$Z_q(s) \equiv \sum_{v=1}^{N_s}|p(v,s)|^q = N_s F_q^q(s)$$

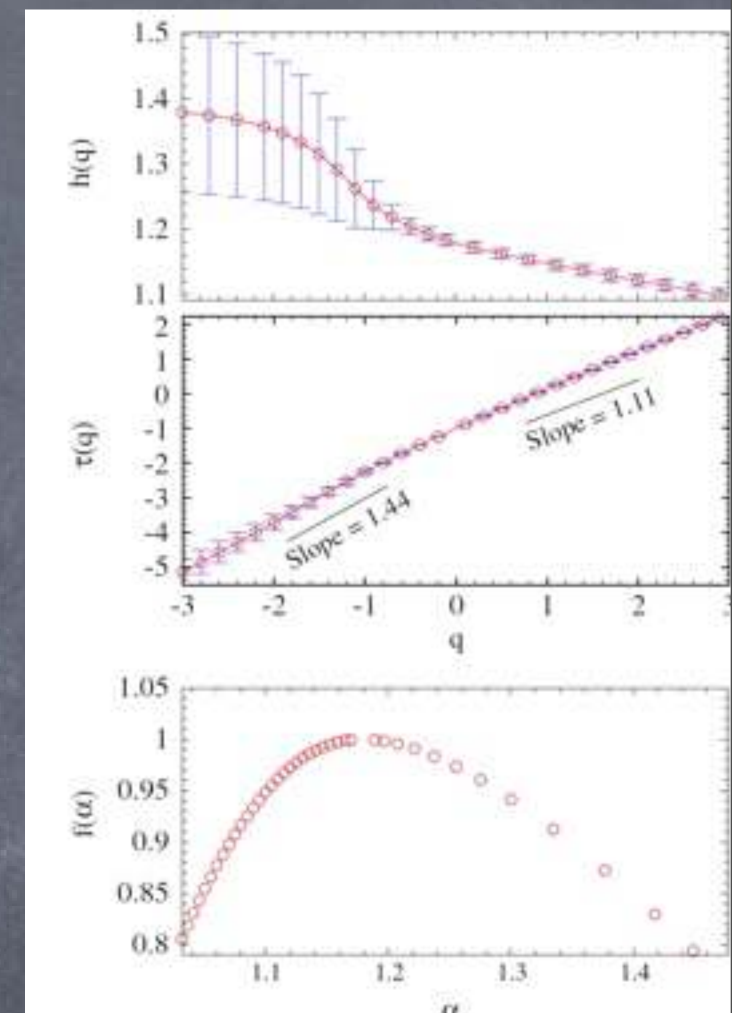
$$\sim \frac{N}{s} s^{qh(q)} \sim s^{qh(q)-1} \sim s^{\tau(q)}$$

$$\tau(q) = qh(q) - 1$$

Free energy and T^{-1}

Singularity spectrum

- A criterion for scaling behavior of measure at each subinterval of time series



$$p(v,s) \sim s^{\alpha_v} \qquad for \quad s \to 0$$

$$PDF \to \mu(\alpha) \sim l^{-f(\alpha)}$$

$$\alpha = \tau'(q)$$

$$\alpha = h(q) + qh'(q)$$

$$f(\alpha) = q\left[\alpha - h(q)\right] + 1$$

$$\Delta\alpha \equiv \alpha(q_{\min}) - \alpha(q_{\max})$$

$$\Delta\alpha \to 0$$

$$f(\alpha = H) = 1$$

A Holder exponent represents monofractal process while the existence of spectrum for Holder exponent demonstrates multifractality nature of time series

Necessary condition for FT

- Integration of series should be finite
- Derivative can be defined for series
- Series should be periodic

In many cases, in discrete measurements, above conditions are not satisfied

Power spectrum

- Fourier or legender transformation of correlation function

$$C_x(i,j) = \langle x(i)x(j) \rangle$$
$$= C_x\left(|i-j|\right) \equiv C_x(\tau) = \langle x(i+\tau)x(i) \rangle$$

$$S_x(\nu) = \frac{1}{2T}\int_{-T}^{T} C_x(\tau)e^{i\omega\tau}d\tau$$

$$\sigma^2 = C_x(0) = \int_{-T}^{T} S_x(\omega) d\omega$$

$$C_x(\tau) = C_x(-\tau)$$

$$S_x(\omega) = A(\omega) + iB(\omega)$$

$$B(\omega) = \frac{1}{2T} \int_{-T}^{T} C_x(\tau) \sin(\omega\tau) d\tau = 0$$

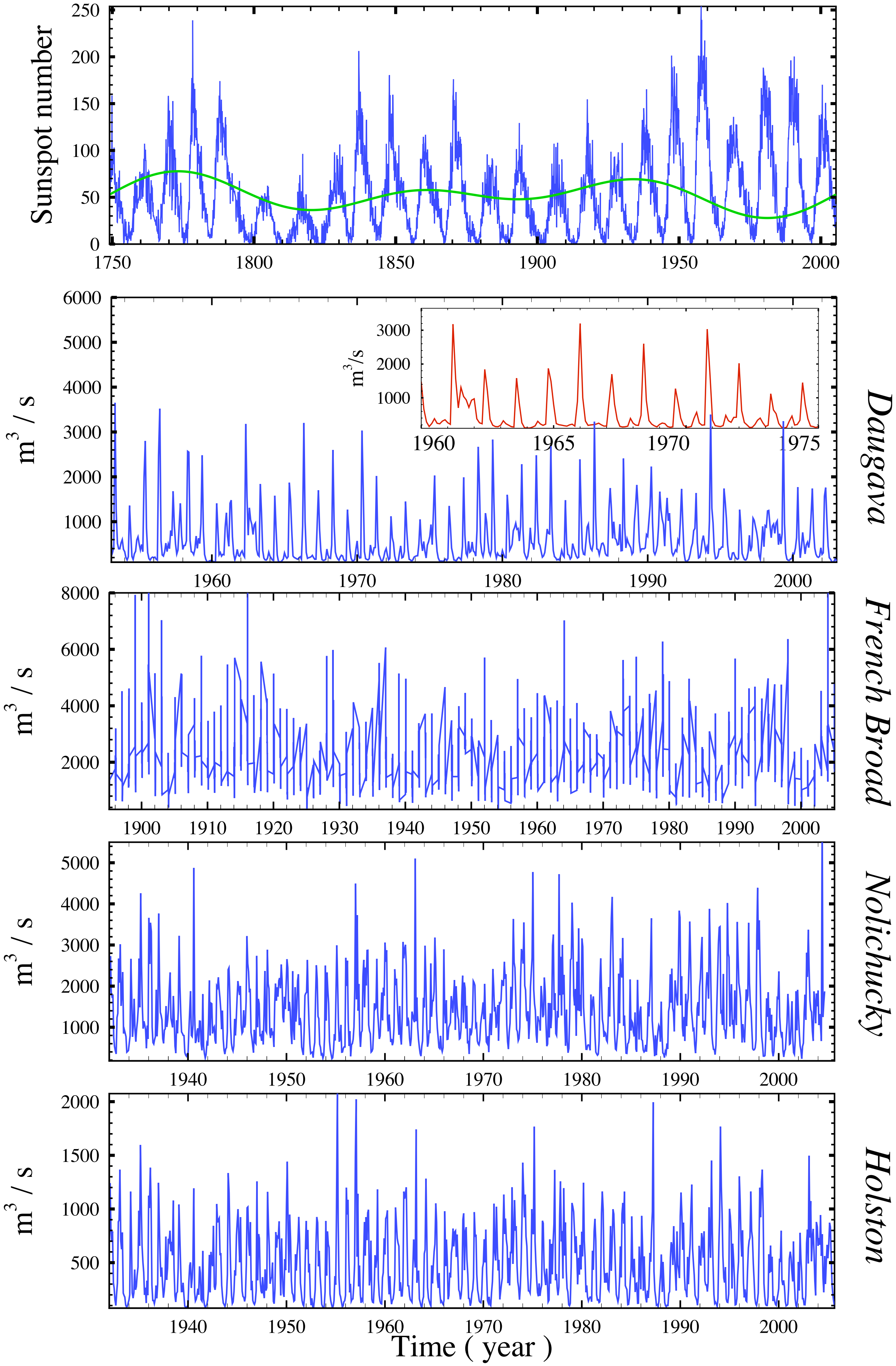
$$S_x(\omega) = | X(\omega) |^2$$

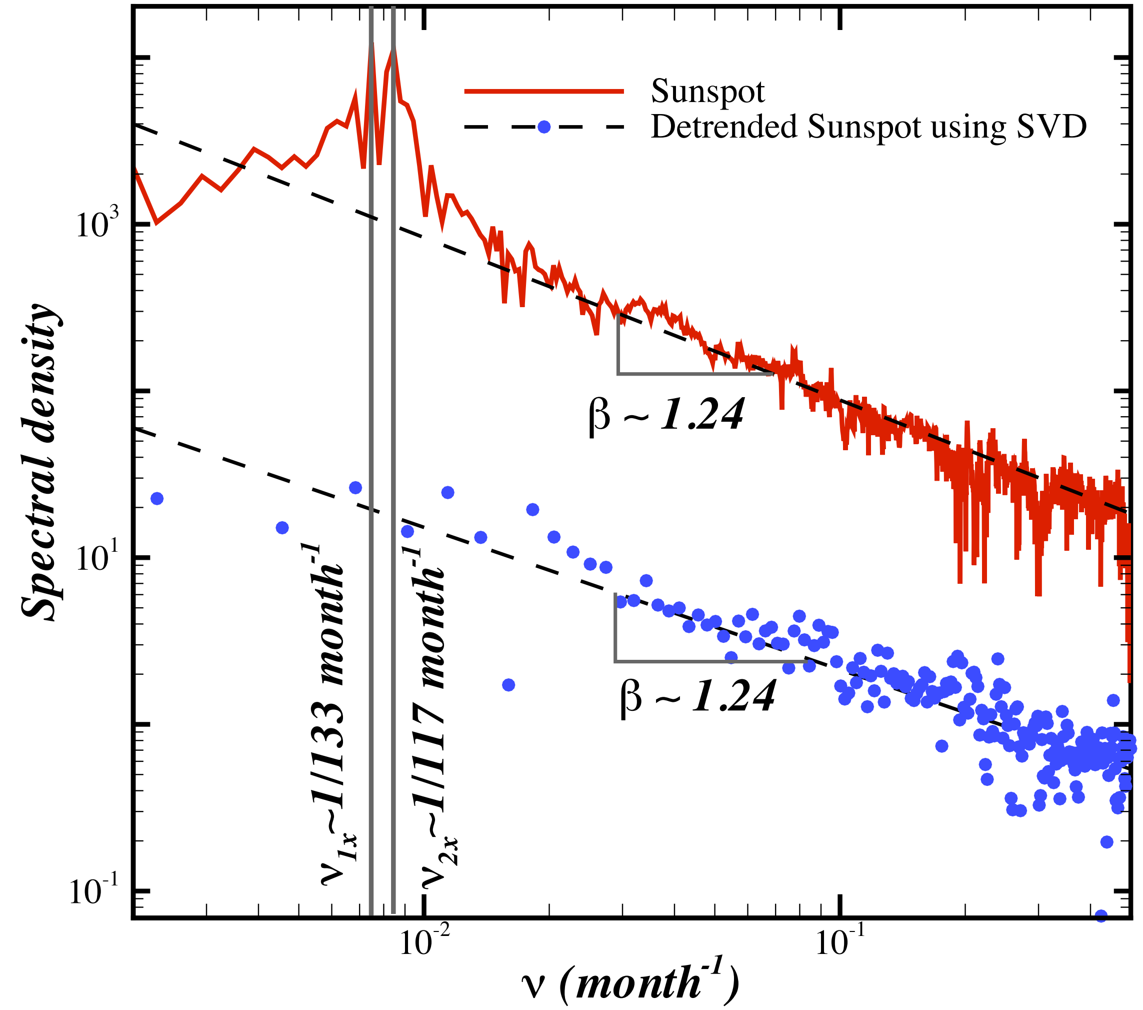
$$S_x(\omega) = \frac{1}{2T} \int_{-T}^{T} C_x(\tau) e^{i\omega\tau} d\tau = \frac{1}{2T} \int_{-T}^{T} \langle x(t) \cdot x(t+\tau) \rangle e^{i\omega\tau} d\tau$$

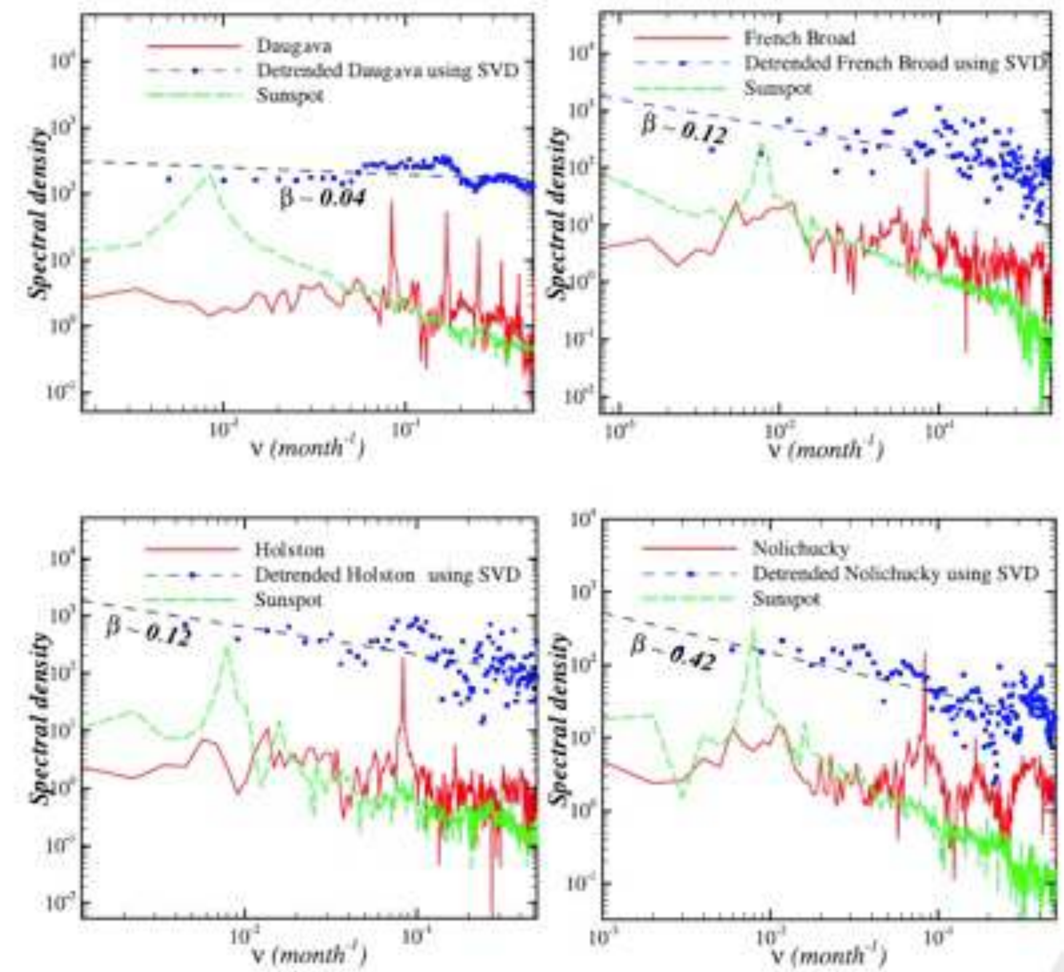
$$= \frac{1}{2T} \int_{-T}^{T} \frac{1}{T} \int_{-T}^{T} x(t) \cdot x(t+\tau) dt \, e^{i\omega\tau} d\tau$$

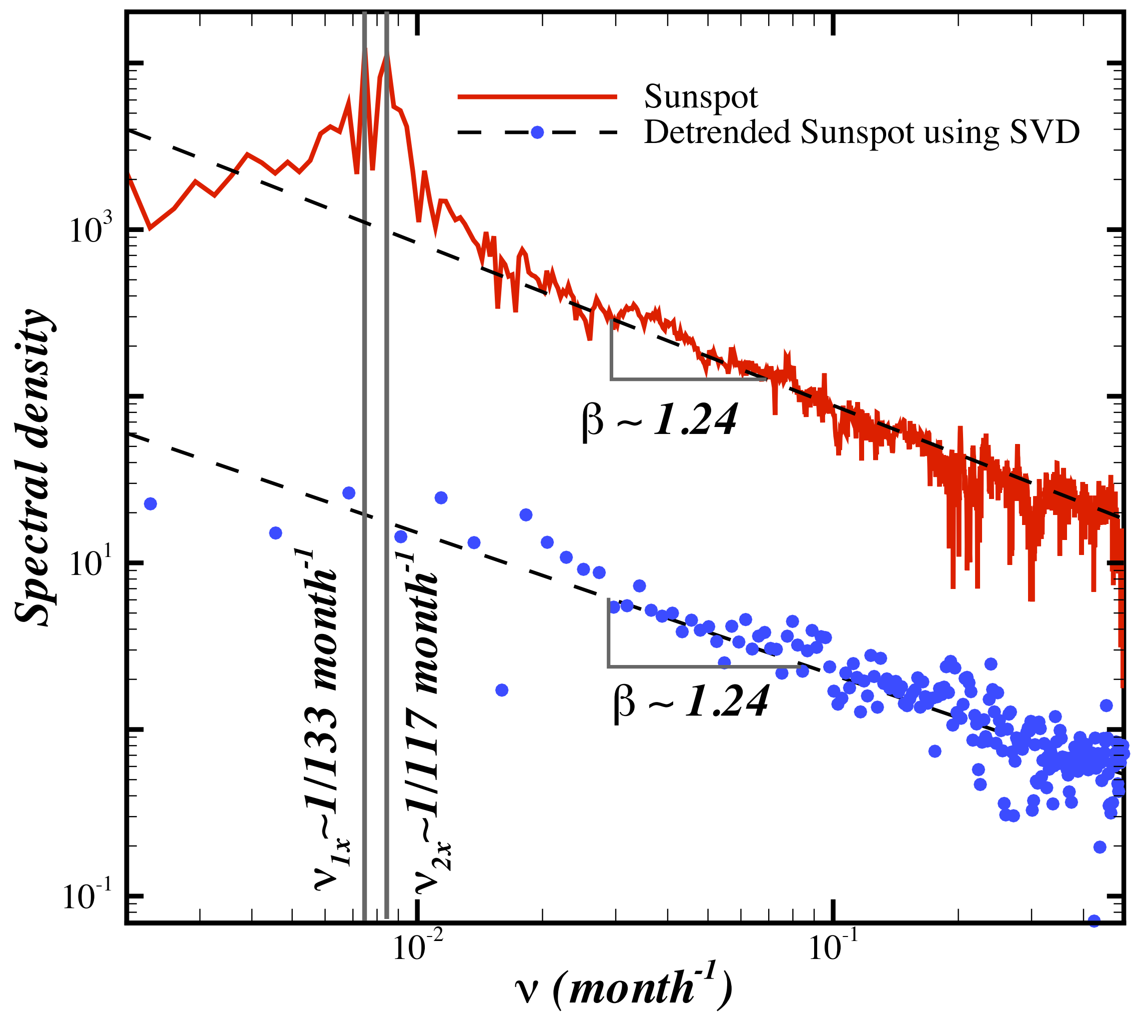
$$= \frac{1}{2T^2} \int_{-T}^{T} \int_{-T}^{T} \left(\int X(\omega') e^{-i\omega' t} d\omega' \right) \left(\int X(\omega'') e^{i\omega''(t+\tau)} d\omega'' \right) dt \, e^{i\omega\tau} d\tau$$

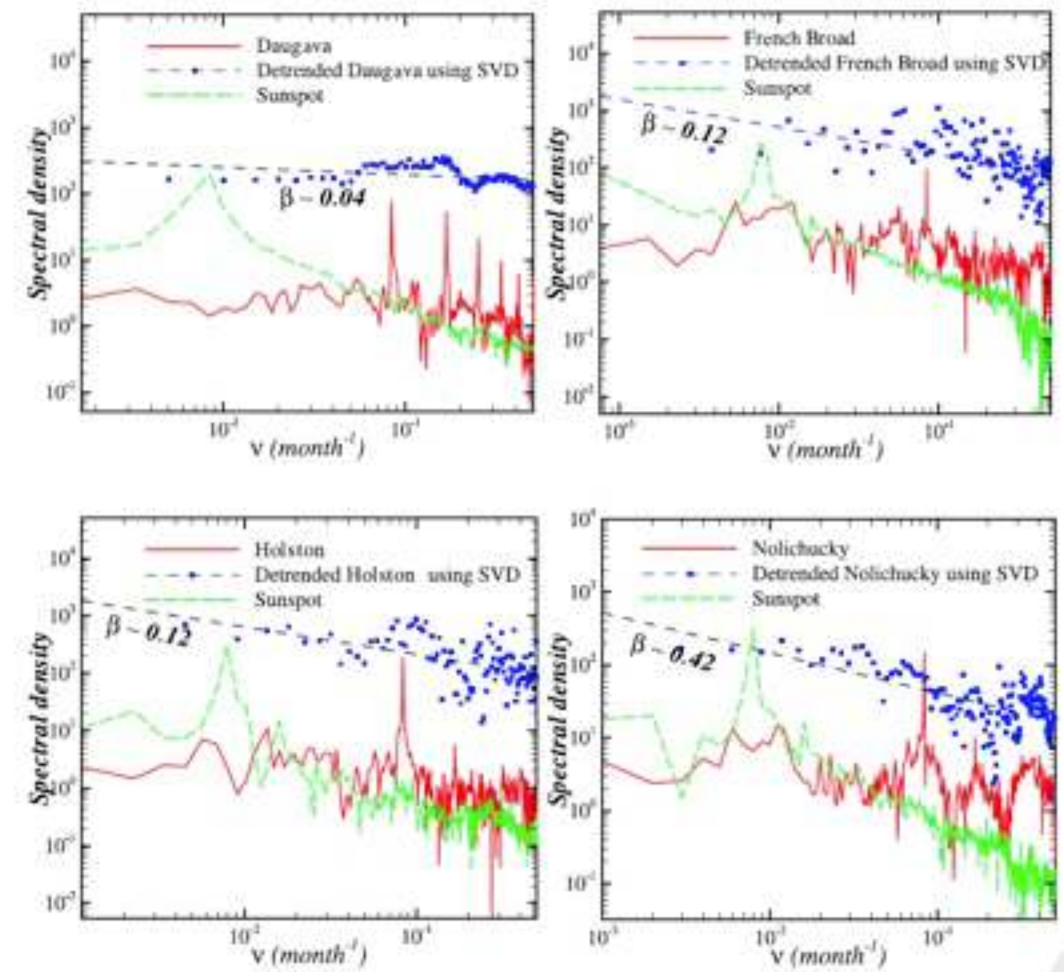
$$= \frac{1}{2T^2} (2\pi)^2 \delta(\omega - \omega'') \delta(\omega' + \omega) X(\omega) X^*(\omega)$$


Sunspot number
m³ / s
Daugava
French Broad
Nolichucky
Holston
Time (year)


Sunspot
Detrended Sunspot using SVD
Spectral density
β ~ 1.24
β ~ 1.24
ν1x~1/133 month-1
ν2x~1/117 month-1
ν (month-1)


Daugava
Detrended Daugava using SVD
Sunspot
β ~ 0.04
French Broad
Detrended French Broad using SVD
Sunspot
β ~ 0.12
Holston
Detrended Holston using SVD
Sunspot
β ~ 0.12
Nolichucky
Detrended Nolichucky using SVD
Sunspot
β ~ 0.42
Spectral density
ν (month-1)


Sunspot
Detrended Sunspot using SVD
Spectral density
β ~ 1.24
β ~ 1.24
ν1x~1/133 month-1
ν2x~1/117 month-1
ν (month-1)


Daugava
Detrended Daugava using SVD
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Detrended Holston using SVD
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β ~ 0.12
Nolichucky
Detrended Nolichucky using SVD
Sunspot
β ~ 0.42
Spectral density
ν (month-1)

Power spectrum exponent

$S(\nu) \sim \nu^{-\beta}$

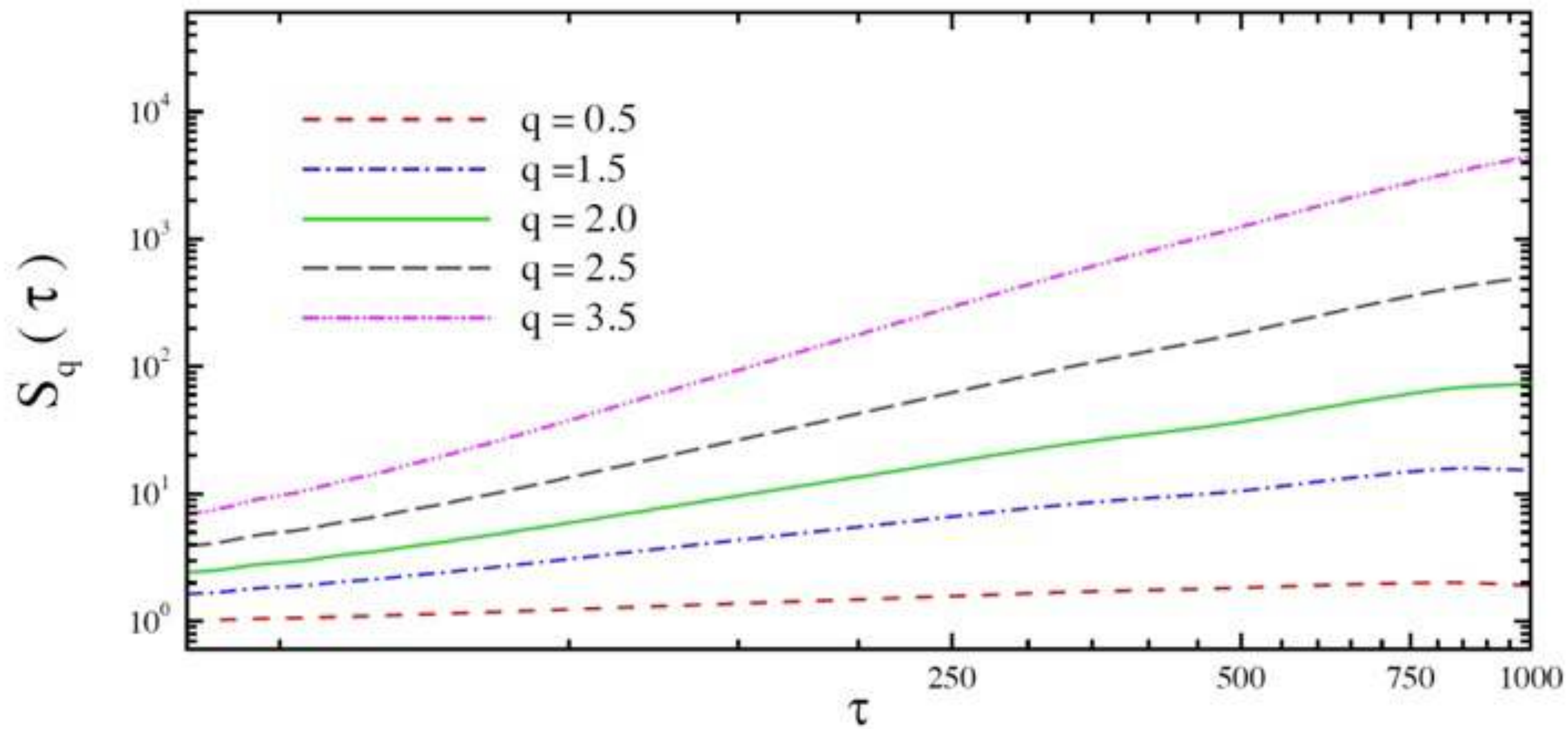
$$S(\nu) \sim \nu^{-\beta} \sim \nu^{-1+\gamma}$$

$$\gamma = 2 - 2H$$

$$\beta = 2H - 1$$ For fGn

$$\beta = 2H + 1$$ For fBm

Extended self-similarity and Hurst exponent

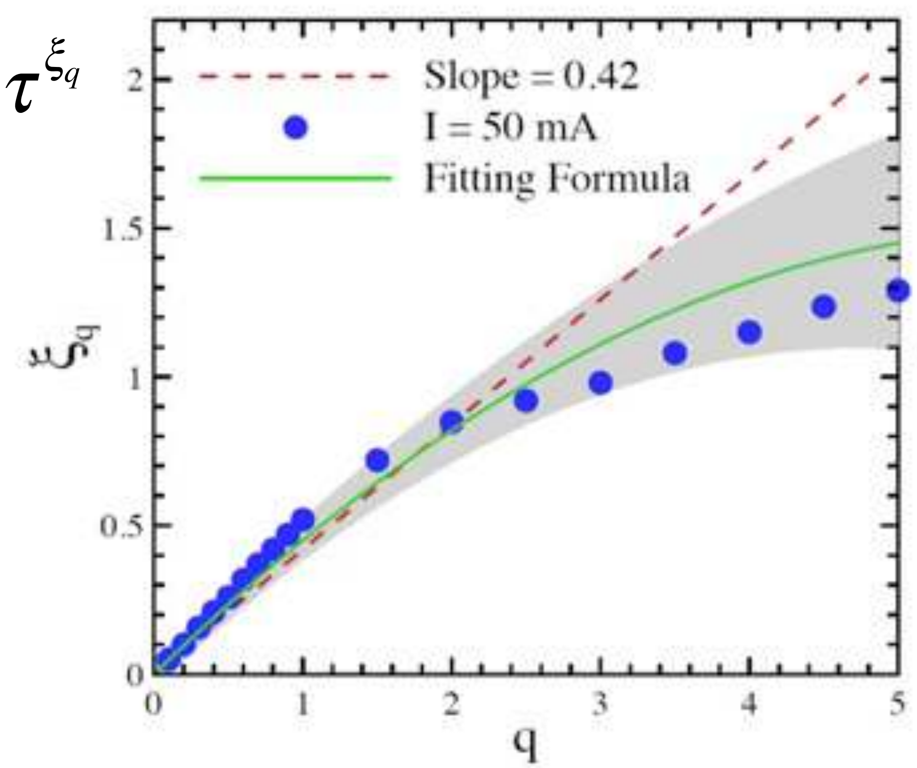


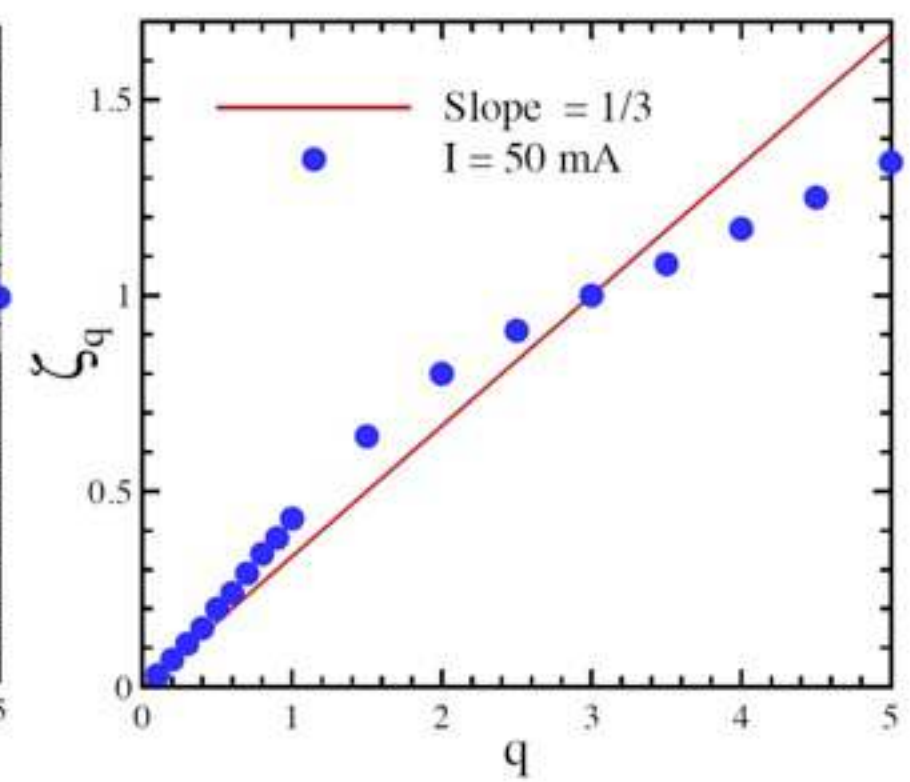
$$S_q(\tau) \equiv \left\langle |x(i+\tau) - x(i)|^q \right\rangle \sim \tau^{\xi_q}$$

$$S_q(\tau) \sim S_s(\tau)^{\zeta_q}$$

$$\xi_q = qH - q(q-1)b$$

$\zeta_q = \dfrac{1}{3}$ For Gaussian data





S. Kimiagar, M. Sadegh Movahed et. al., arXiv:0710.5270

$D(fractal - dimention)$	H_{fBm}	H_{fGn}	β	$h(q=\text{۲})$	
$-$	$\text{۲} - H_{fBm}$	$-$	$\frac{\text{۵}-\beta}{\text{۲}}$	$\text{۳} - h(q=\text{۲})$	D
$\text{۲} - D$	$-$	$-$	$\frac{\beta-\text{۱}}{\text{۲}}$	$h(q=\text{۲}) - \text{۱}$	H_{fBm}
$-$	$-$	$-$	$\frac{\beta+\text{۱}}{\text{۲}}$	$h(q=\text{۲})$	H_{fGn}
$\text{۵} - \text{۲}D$	$\text{۲}H_{fBm} + \text{۱}$	$\text{۲}H_{fGn} - \text{۱}$	$-$	$\text{۲}h(q=\text{۲}) - \text{۱}$	β
$\text{۳} - D$	$H_{fBm} + \text{۱}$	H_{fGn}	$\frac{\beta+\text{۱}}{\text{۲}}$	$-$	$h(q=\text{۲})$

q	$\tau(q)$	$\alpha = -\frac{d\tau(q)}{dq}$	$f = q\alpha + \tau(q)$
$q \to -\infty$	$-q\alpha_{\max}$	$\alpha_{\max} = -\ln\mu_- / \ln\delta$	0
$q = 0$	D	α_0	D
$q = 1$	0	$\alpha_1 = -S(\delta) / \ln\delta$	α_1
$q \to +\infty$	$-q\alpha_{\min}$	$\alpha_{\min} = -\ln\mu_+ / \ln\delta$	0

Fractals: Jens Feder 1988

Multifractality

A: $h(q)$ depends on "q"

B: There is a spectrum for holder exponent

C: There are various slopes for $\tau(q)$ in different scales

What are the sources?

1) Multifractality due to a fatness of PDF

2) Multifractality due to different correlations in small and large scales

$$F_q(s)/F_q^{\mathrm{shuf}}(s) \sim s^{h(q)-h_{\mathrm{shuf}}(q)} = s^{h_{\mathrm{cor}}(q)},$$

$$F_q(s)/F_q^{\mathrm{sur}}(s) \sim s^{h(q)-h_{\mathrm{sur}}(q)} = s^{h_{\mathrm{PDF}}(q)}.$$

$h_{\mathrm{cor}}(q) = 0$ For Fatness

$h_{\mathrm{PDF}}(q) = 0$ For correlation

What are shuf" and "sur"?
Actually there are the abbreviation of
Shuffled and Surrogate data set

Surrogate method

(i) Computing the discrete Fourier transform (DFT) coefficients of the series

$$\mathcal{F}^2\{x(t)\} \equiv |X(\nu)|^2 = |X(k)|^2 = \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(t_n) \mathrm{e}^{\mathrm{i}2\pi nk/N} \right|^2 \tag{9}$$

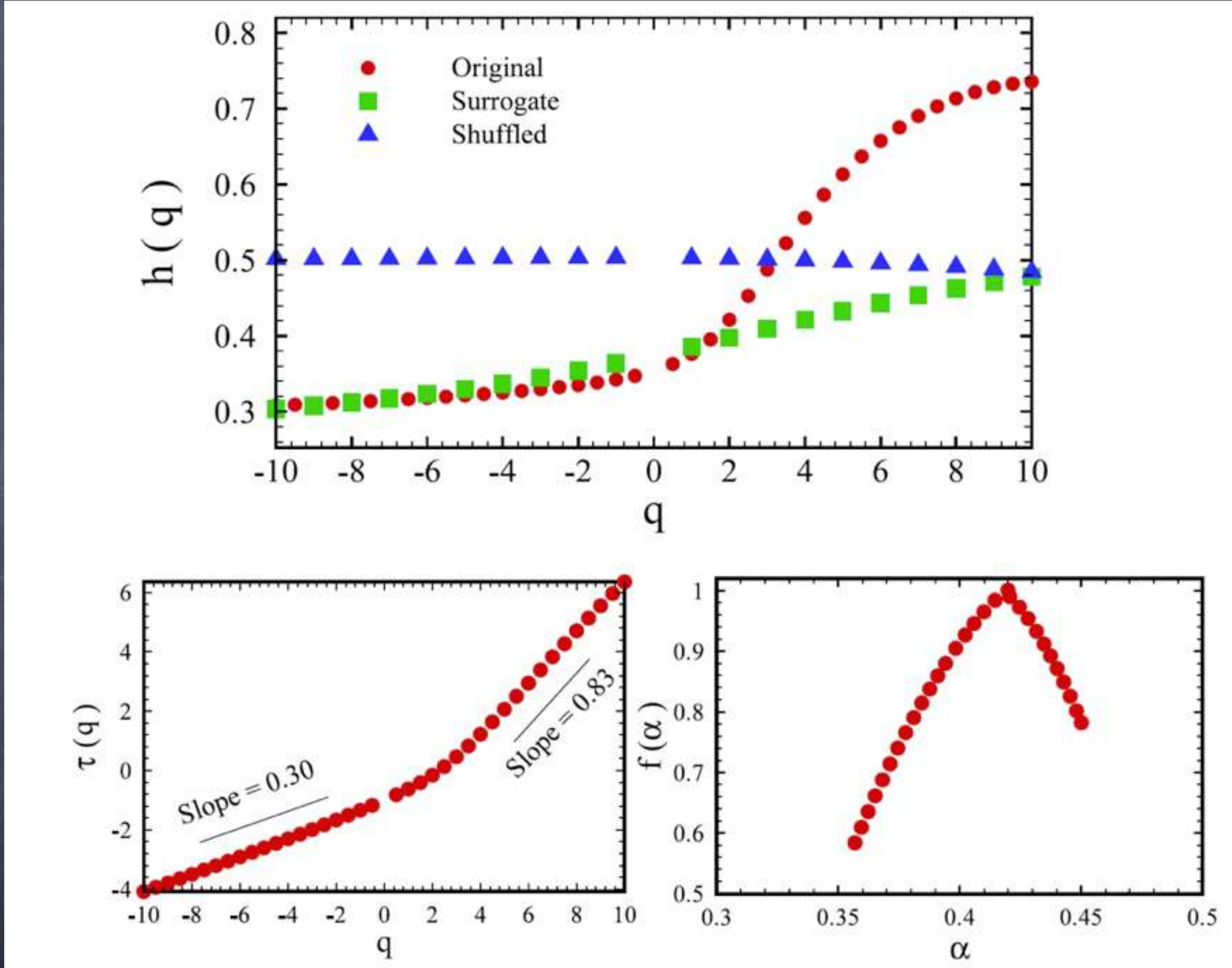
where $\nu = k/N\Delta t$ and Δt is the step of digitization in the experimental setup.

(ii) Multiplying the DFT coefficients of the series by a set of pseudo-independent, uniformly distributed $\phi(\nu)$ quantities in the range $[0, 2\pi)$:

$$\tilde{X}(\nu) = X(\nu)\mathrm{e}^{\mathrm{i}\phi(\nu)}. \tag{10}$$

(iii) The surrogate data set is given by the inverse DFT as

$$\mathcal{F}^{-1}\{\tilde{X}(\nu)\} \equiv \tilde{x}(t_n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |X_k| \, \mathrm{e}^{\mathrm{i}\phi(k)} \mathrm{e}^{-\mathrm{i}2\pi nk/N}. \tag{11}$$



S. Kimiagar, M. Sadegh Movahed et. al., JSTAT P03020 (2009)

MF-DFA in higher dimension

In many cases, one encounters with self-similar of self-affine surface which is denoted by a two dimensional array X(i,j). For this case the MF-DFA has the following steps:

Step I: Suppose

$$x(i,j), \quad \begin{cases} i = 1,\ldots,M \\ j = 1,\ldots,N \end{cases}$$

$$M_s = \mathrm{int}\left(\frac{N}{s}\right)$$

$$N_s = \mathrm{int}\left(\frac{M}{s}\right)$$

$$x_{v,w}(i,j) = x(l_1 + i, l_2 + j) \quad 1 \le i,j \le s \qquad \begin{cases} l_1 = (v-1)s \\ l_2 = (w-1)s \end{cases}$$

Step II: For each non-overlapping segment, the cumulative sum is calculated by:

$$Y_{v,w}(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} x_{v,w}(k,l) \qquad 1 \le i,j \le s$$

Step III: The trend of constructed cumulative arrays such as:

$$u_{v,w}(i,j) = a_{v,w} i + b_{v,w} j + c_{v,w}$$

$$u_{v,w}(i,j) = a_{v,w} i^2 + b_{v,w} j^2 + c_{v,w}$$

$$u_{v,w}(i,j) = a_{v,w} ij + b_{v,w} i + c_{v,w} j + d_{v,w}$$

$$u_{v,w}(i,j) = a_{v,w} i^2 + b_{v,w} j^2 + c_{v,w} i + d_{v,w} j + e$$

$$u_{v,w}(i,j) = a_{v,w} i^2 + b_{v,w} j^2 + c_{v,w} ij + d_{v,w} i + e_{v,w} j + f_{v,w}$$

Step IV: For each non-overlapping segment, the cumulative sum is calculated by:

$$\varepsilon_{v,w}(i,j) = Y_{v,w}(i,j) - u_{v,w}(i,j)$$

$$F^2_{v,w}(s) = \frac{1}{s^2} \sum_{i=1}^{s} \sum_{j}^{s} \varepsilon_{v,w}(i,j)^2$$

Step V: By averaging over all segments as:

$$F_q(s) = \left[\frac{1}{N_s M_s} \sum_{v=1}^{N_s} \sum_{w=1}^{M_s} \left\{ F^2_{v,w}(s) \right\}^{q/2} \right]^{1/q}$$

$$F_q(s) = \mathrm{A} \times s^{h(q)} \qquad \begin{cases} s_{\min} \approx 6 \\ s_{\max} \approx \min(M,N)/4 \end{cases}$$

$$Y(i) = \sum_{k=۱}^{i} \sum_{l=۱}^{j} \left[x(k,l) - \langle x \rangle\right]^{۲}$$

$$y_\nu(i,j) = a_\nu + b_\nu i + c_\nu j$$

$$\begin{aligned}
b_\nu &= \frac{\sum_{i,j=۱}^{s,m} Y(i,j)i - \frac{۱}{s\times m}\sum_{i,j=۱}^{s,m} Y(i,j)\sum_{i,j=۱}^{sm} i}{\sum_{i,j=۱}^{s,m} i^{۲} - \frac{۱}{s\times m}\left[\sum_{i,j=۱}^{s,m} i\right]^{۲}}, \\
&\simeq \frac{\sum_{i,j=۱}^{s,m} Y(i,j)i - \frac{s}{۲}\sum_{i,j=۱}^{s,m} Y(i,j)}{m \times s^{۳}/۱۲}
\end{aligned}$$

$$\begin{aligned}
c_\nu &= \frac{\sum_{i,j=۱}^{s,m} Y(i,j)j - \frac{۱}{s\times m}\sum_{i,j=۱}^{s,m} Y(i,j)\sum_{i,j=۱}^{sm} j}{\sum_{i,j=۱}^{s,m} j^{۲} - \frac{۱}{s\times m}\left[\sum_{i,j=۱}^{s,m} j\right]^{۲}}, \\
&\simeq \frac{\sum_{i,j=۱}^{s,m} Y(i,j)j - \frac{m}{۲}\sum_{i,j=۱}^{s,m} Y(i,j)}{s \times m^{۳}/۱۲}
\end{aligned}$$

$$\begin{aligned} a_\nu &= \frac{۱}{s \times m} \sum_{i=۱}^{s} \sum_{j=۱}^{m} Y(i,j) - \frac{b_\nu}{s \times m} \sum_{i=۱}^{s} \sum_{j=۱}^{m} i - \frac{c_\nu}{s \times m} \sum_{i=۱}^{s} \sum_{j=۱}^{m} j \\ &\simeq \frac{۱}{s \times m} \sum_{i,j=۱}^{s,m} Y(i,j) - \frac{b_\nu s}{۲} - \frac{c_\nu m}{۲}, \end{aligned}$$

$$F^{۲}(s;\nu) = \frac{۱}{sm} \sum_{i=۱}^{s} \sum_{j=۱}^{m} [Y_\nu(i,j) - y_\nu(i,j)]^{۲}$$

$$\begin{aligned}
\left\langle \left[F^{۲}(s,m;\nu) \right] \right\rangle &= \left\langle \frac{۱}{s\times m} \sum_{i,j=۱}^{s,m} \left[Y(i,j) - a - bi - cj \right]^{۲} \right\rangle \\
&\simeq \left\langle \frac{۱}{s\times m} \sum_{i,j=۱}^{s,m} Y(i,j)^{۲} \right\rangle + \left\langle a^{۲} \right\rangle + \frac{s^{۲}}{۳} \left\langle b^{۲} \right\rangle + \frac{m^{۲}}{۳} \left\langle c^{۲} \right\rangle \\
&\quad -۲ \left\langle \frac{a}{s\times m} \sum_{i,j=۱}^{s,m} Y(i,j) \right\rangle - ۲ \left\langle \frac{b}{s\times m} \sum_{i,j=۱}^{sm} iY(i,j) \right\rangle \\
&\quad -۲ \left\langle \frac{c}{s\times m} \sum_{i,j=۱}^{sm} jY(i,j) \right\rangle + s\langle ab\rangle + m\langle ac\rangle + \frac{s\times m}{۲} \langle bc\rangle
\end{aligned}$$

$$Y(i,j) = (ij)^H x$$

$$\begin{aligned}
Y(i,j) - Y(k,l) &= Y(i,l) + Y(k,j) + |i-k|^H |j-l|^H x \\
&= \left[(il)^H + (kj)^H + |i-k|^H |j-l|^H \right] x,
\end{aligned}$$

$$\langle [Y(i,j) - Y(k,l)]^{۲} \rangle = \sigma^{۲} \left[(il)^H + (kj)^H + |i-k|^H |j-l|^H \right]^{۲}$$

$$\sigma^{۲} = \left\langle x(i,j)^{۲} \right\rangle$$

$$\left\langle Y(i,j)^{۲} \right\rangle = \sigma^{۲} (ij)^{۲H}$$

$$\langle Y(i,j) Y(k,l) \rangle = \frac{\sigma^{۲}}{۲} [(ij)^{۲H} + (kl)^{۲H} - (ik)^{۲H} - (jl)^{۲H}$$

$$-۲ |i-k|^H |j-l|^H \left[(il)^H + (kj)^H \right]$$

$$-|i-k|^{۲H} |j-l|^{۲H} - ۲ (ijkl)^H],$$

$$\left\langle \left[F^{\text{٢}}(s, m; \nu) \right] \right\rangle_{\nu} = C_H (s \times m)^{\text{٢}H}$$

$$
\begin{aligned}
C_H \;=\; & \frac{۱}{۲}\sigma^{۲}\Big[\frac{۲۰ + H\left(۸۰ + H\left[۱۴۳ + H(۲ + H)(۷۹ + ۲H(۶ + H))\right]\right)}{(۱ + H)^{۴}(۲ + H)^{۲}(۱ + ۲H)^{۲}} \\
& + \Big\{ -\frac{۱۲(۲\Gamma[۲ + H]^{۲} + \Gamma[۳ + ۲H])}{(۱ + H)^{۳}(۲ + H)\Gamma[۳ + ۲H]\Gamma[۴ + ۲H]^{۲}} \\
& \times \Big\{ ۲\Gamma[۴ + ۲H]\left(\Gamma[۲ + H]\left\{۲H^{۳}\Gamma[H] + (۳ + ۵H)\Gamma[۱ + H] + \Gamma[۲ + H]\right\} + (۳ + H)\Gamma[۳ + ۲H]\right) \\
& + \Gamma[۲ + H]^{۲}\Gamma[۵ + ۲H]\Big\} + \left(\frac{۱}{(۱ + H)^{۲}} + \frac{۲\Gamma[۱ + H]^{۲}}{\Gamma[۳ + ۲H]}\right) \\
& \times \Big(۷\left[\frac{۱}{(۱ + H)^{۲}} + \frac{۲\Gamma[۱ + H]^{۲}}{\Gamma[۳ + ۲H]}\right] + \frac{۴۸}{(۱ + H)(۲ + H)\Gamma[۵ + ۲H]^{۲}} \\
& \times \left(۲\Gamma[۳ + H]^{۲}\Gamma[۴ + ۲H] + \left(۲(۱ + H)(۲ + H)\Gamma[۲ + H]^{۲} + \Gamma[۴ + ۲H]\right)\Gamma[۵ + ۲H]\right)\Big)\Big\}\Big]
\end{aligned}
$$

$$
\Gamma(x) \equiv (x - ۱)! = \int_{۰}^{\infty} t^{x-۱} e^{-t}\, dt
$$

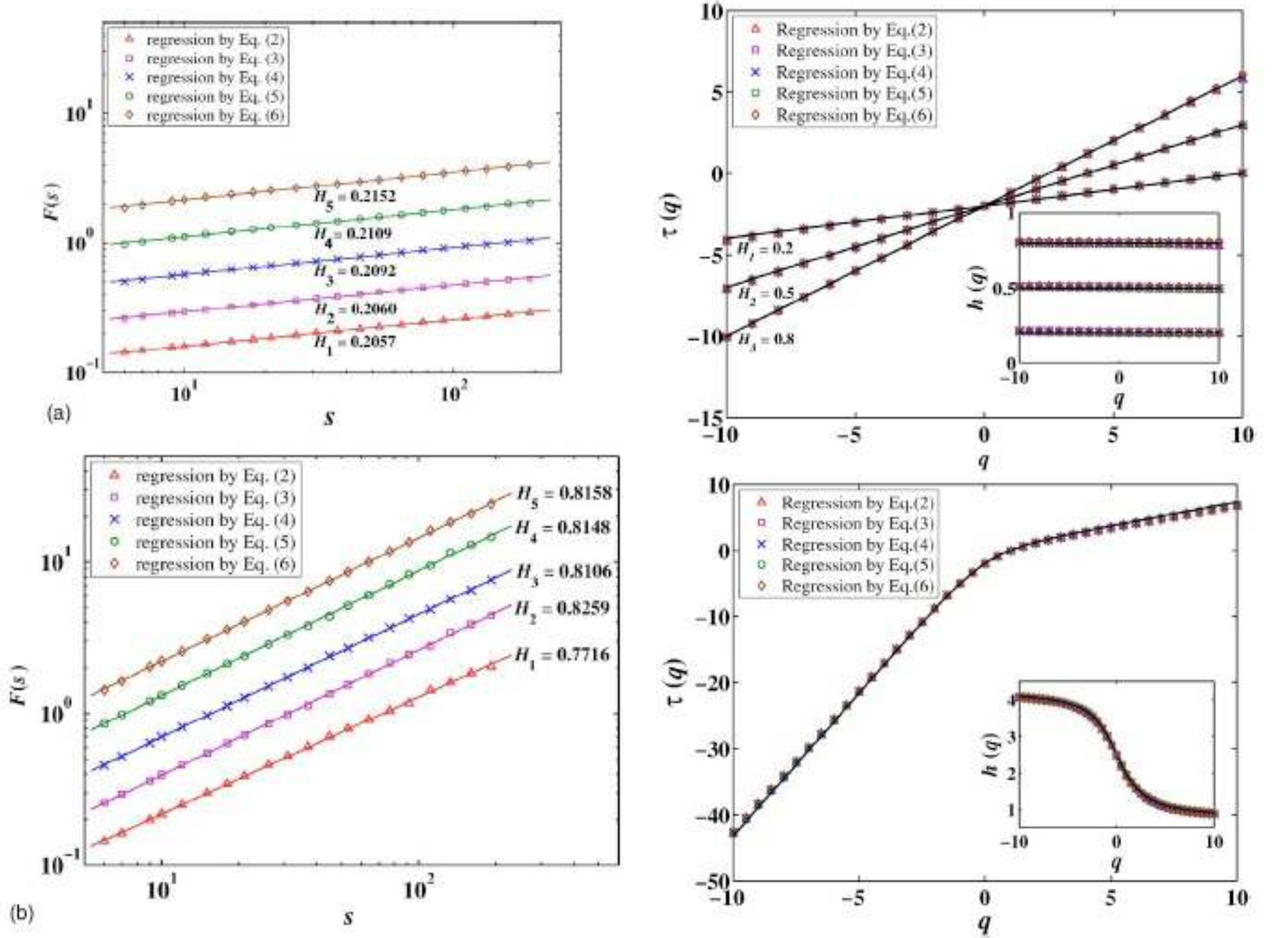
Some important exponents

$$\tau(q) = qh(q) - d_f$$

$$D_f = 3 - H$$

$$f(x) = \mu(x) \otimes |x|^{-(1-H)} \qquad H \in (0,1)$$

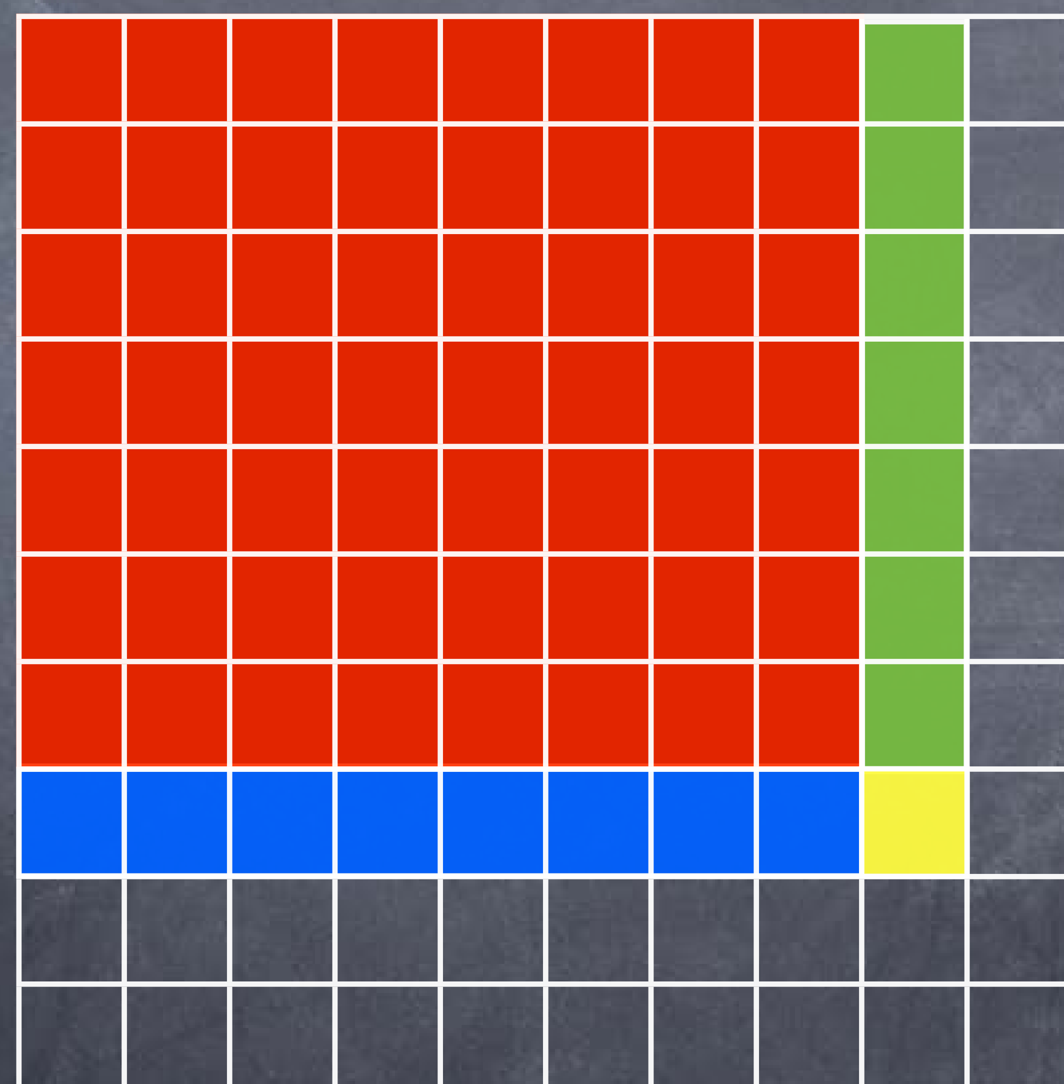
$$\tau(q) = q(1+H) - 1 - \log_2\left[p^q + (1-p)^q \right]$$



Gao-Feng Gu and Wei-Xing Zhou, PHYSICAL REVIEW E **74**, 061104 (2006)

More about cumulative sum

$$X(i,j) = X_{v,w}(i-1,j-1) + \sum_{k=1}^{i-1} x(k,j) + \sum_{l=1}^{j-1} x(i,l) + x(i,j)$$



More about cumulative sum

$$X(l_v + i, l_w + j) = X_{v,w}(i,j) + \sum_{k=1}^{l_v}\sum_{l=1}^{l_w} x(k,l) + \sum_{k=1}^{l_v}\sum_{l=l_w+1}^{l_w+j} x(k,l) + \sum_{k=l_v+1}^{l_v+i}\sum_{l=1}^{l_w} x(k,l)$$

$s = 4$

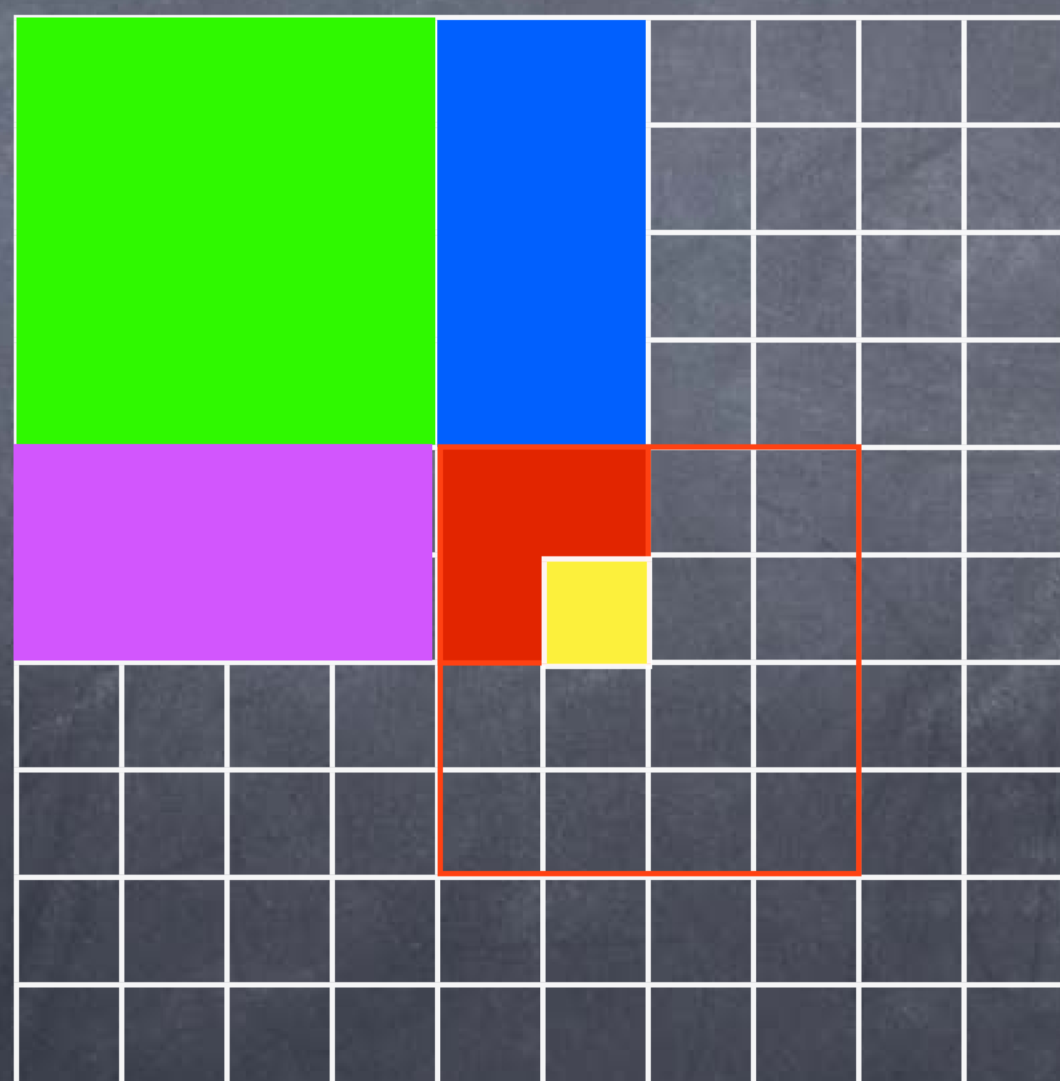
$i = j = 2$

$v = 2$

$w = 2$

$l_2 = 4$

$l_2 = 4$



Multifractal Detrended cross-correlation (MF-DCCA)

- Step I: Consider two time series as:

$$\{x(i)\} \quad \{y(i)\} \quad i = 1,2,\ldots,N$$

$$M_s = \text{int}\left(\frac{N}{s}\right)$$

- Step II: Construct profile and trend functions. Polynomials or based on empirical mode decomposition (EMD, non-parametric)

$$X_v(k) = \sum_{i=1}^{k} x(l_v + i) \qquad l_v = (v-1)s$$

$$Y_v(k) = \sum_{i=1}^{k} y(l_v + i)$$

$$F(s,v) = \frac{1}{s}\sum_{i}^{s}\left\{Y[(v-1)s + i - y_v(i)\right\} \times \left\{X[(v-1)s + i - x_v(i)\right\} \qquad v = 1,\ldots,M_s$$

$$F(s,v) = \frac{1}{s}\sum_{i}^{s}\left\{Y[N-(v-1)s + i - y_v(i)\right\} \times \left\{X[N-(v-1)s + i - x_v(i)\right\} \qquad v = M_s + 1,\ldots,2M_s$$

B. Podobnik and H. Eugene Stanley, PRL 100, 084102 (2008)
Wei-Xing Zhou, PRE 77, 066211 (2008)

- Step IV: Averaging over all segments as:

$$F_q(s) = \left\{ \frac{1}{M_s} \sum_{v=1}^{M_s} [F(s,v)]^{q/2} \right\}^{1/q}$$

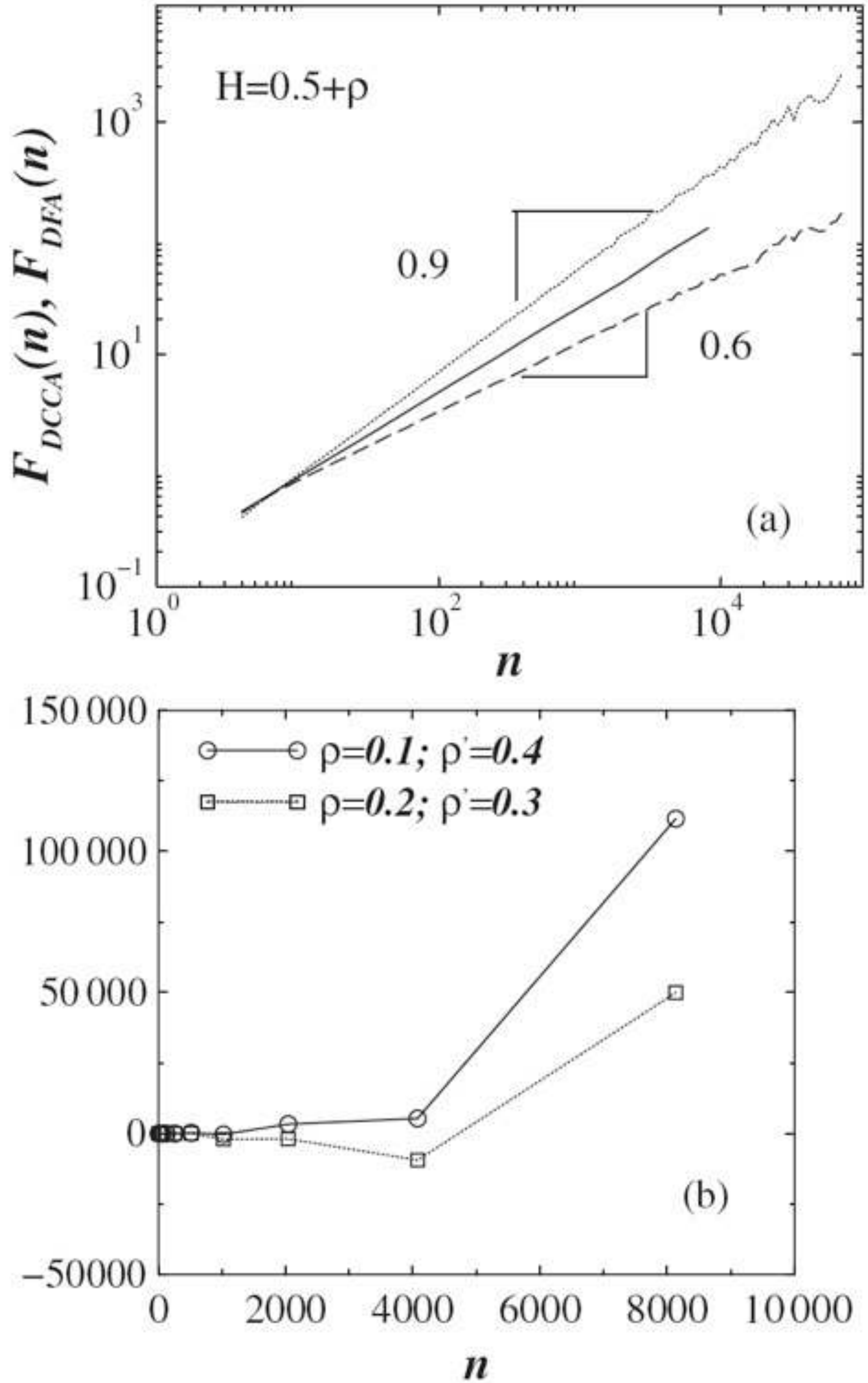
$$F_0(s) = \exp\left(\frac{1}{2M_s} \sum_{v=1}^{M_s} \ln[F(s,v)] \right)$$

- Step V: Demanding a scaling relation according to:

$$F_q(s) \sim s^{\lambda(q)}$$

- If two underlying series to be equal so one finds nothing except the Hurst exponent:

$$F_q(s) \sim s^{h(q)}$$



B. Podobnik and H. Eugene Stanley,
PRL 100, 084102 (2008)

2D version of MF-DCCA

$$x(i,j) \qquad y(i,j) \qquad i=1,\ldots,M \qquad j=1,\ldots,N$$

$$M_s = \operatorname{int}\left(\frac{M}{s}\right) \qquad N_s = \operatorname{int}\left(\frac{N}{s}\right)$$

$$X_{v,w}(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} x_{v,w}(k,l)$$

$$Y_{v,w}(i,j) = \sum_{k=1}^{i}\sum_{l=1}^{j} y_{v,w}(k,l)$$

$$F_{v,w}(s) = \frac{1}{s^2}\sum_{i=1}^{s}\sum_{j=1}^{s}\left[X_{v,w}(i,j) - \tilde{X}_{v,w}(i,j)\right]\left[Y_{v,w}(i,j) - \tilde{Y}_{v,w}(i,j)\right]$$

$$F_q(s) = \left(\frac{1}{M_s N_s}\sum_{v=1}^{M_s}\sum_{w=1}^{N_s}\left[F_{v,w}(s)\right]^{q/2}\right)^{1/q}$$

$$F_0(s) = \exp\left(\frac{1}{2M_s N_s}\sum_{v=1}^{M_s}\sum_{w=1}^{N_s}\ln\left[F_{v,w}(s)\right]\right)$$

$$F_q(s) \sim s^{-\lambda(q)}$$

Wei-Xing Zhou, PRE 77, 066211 2008

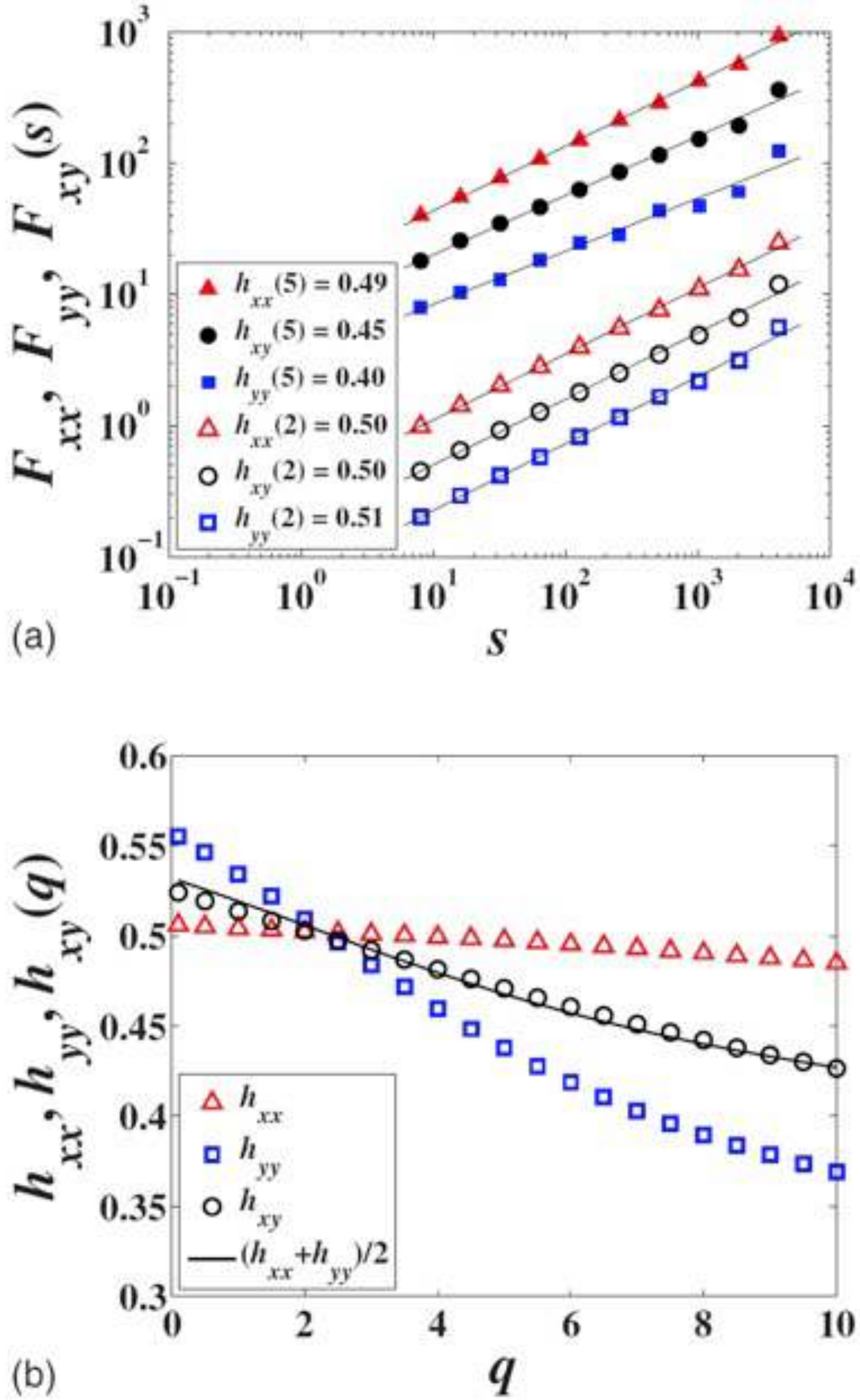


FIG. 2. (Color online) Multifractal nature of the power-law cross correlations of two MRWs. (a) Power-law scaling in F_{xy}, F_{xx}, and F_{yy} with respect to s for $q=2$ and 5; (b) power-law exponents h_{xy}, h_{xx}, and h_{yy}.

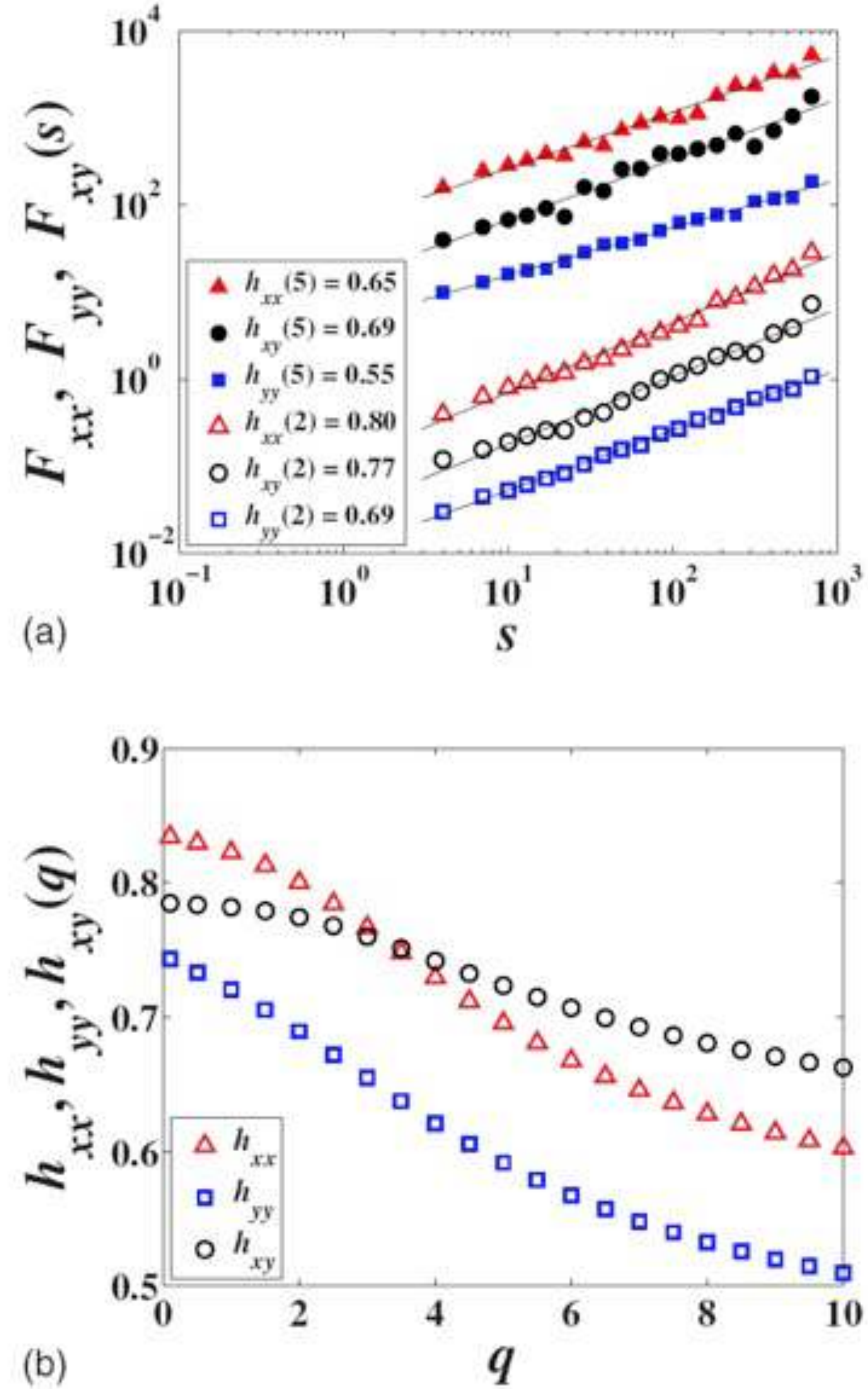


FIG. 3. (Color online) Multifractal nature of the power-law cross correlations of the absolute values of daily price changes for DJIA and NASDAQ indices in the period from July 1993 to November 2003. (a) Power-law scaling in F_{xy}, F_{xx}, and F_{yy} with respect to s for $q=2$ and 5. The scaling range is the same as in Ref. [15]. (b) Dependence of the power-law exponents h_{xy}, h_{xx}, and h_{yy} as nonlinear functions of q, indicating the presence of multifractality. There is no clear relation between these exponents.

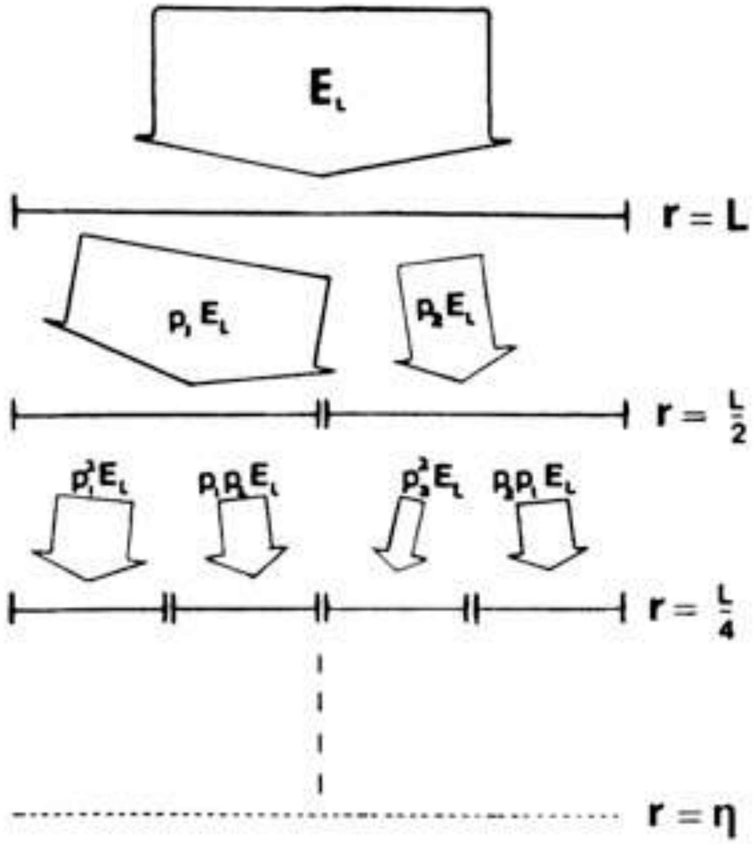


FIG. 3. One-dimensional version of a cascade model of eddies, each breaking down into two new ones. The flux of kinetic energy to smaller scales is divided into nonequal fractions p_1 and p_2. This cascade terminates when the eddies are of the size of the Kolmogorov scale, η.

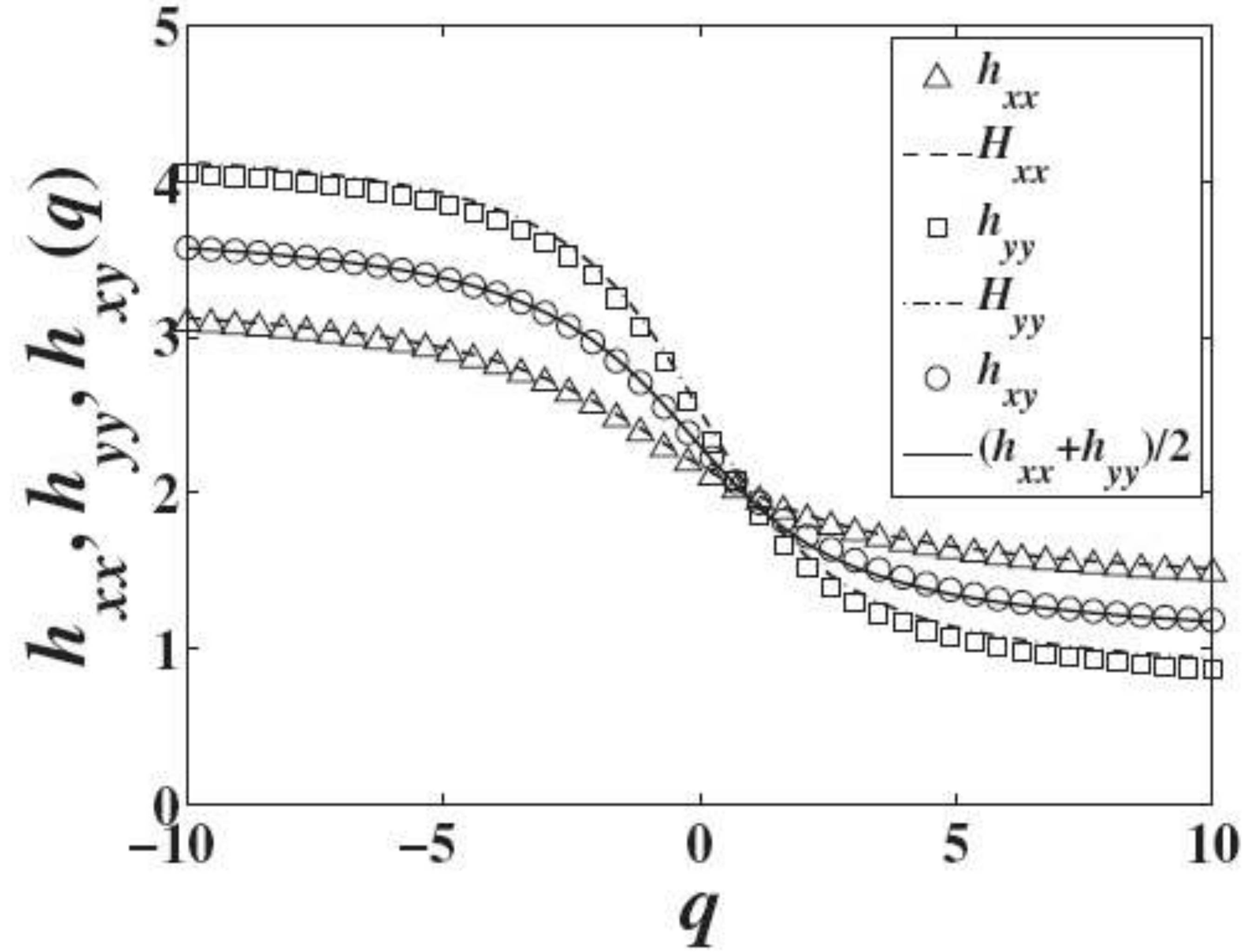


FIG. 4. Multifractal detrended cross-correlation analysis of two cross-correlated synthetic binomial measures from the p model. The size of each multifractal is 4096×4096 and the cross-correlation coefficient is 0.48. The numerical exponents $h_{xx}(q)$ and $h_{yy}(q)$ obtained from the multifractal detrended fluctuation analysis of X and Y are located approximately on the analytical curves $H_{xx}(q)$ and $H_{yy}(q)$. This example illustrates the relation $h_{xy}(q) = [h_{xx}(q)+h_{yy}(q)]/2$.

1) B. B. Mandelbrot, J. Fluid Mech. **62**, 331 1974.
2) C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 1987.
3)E. A. Novikov, Phys. Fluids A **2**, 814 1990
4) C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 1991.

$$H_{zz}(q) = [2 - \log_2(p_{11}^q + p_{12}^q + p_{21}^q + p_{22}^q)]/q,$$

Cross-correlation exponent for stationary series

$$x(i,j) \qquad y(i,j) \qquad i=1,\ldots,M \qquad j=1,\ldots,N$$

$$\mu_x = \frac{1}{N}\sum_{i=1}^{N} x(i) \qquad \sigma_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left[x(i)-\mu_x\right]^2$$

$$\mu_y = \frac{1}{N}\sum_{i=1}^{N} y(i) \qquad \sigma_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left[y(i)-\mu_y\right]^2$$

$$C_x(\tau) = \frac{\left\langle (x(i+\tau)-\mu_x)(x(i)-\mu_x)\right\rangle}{\sigma_x^2} \sim \tau^{-\gamma_x}$$

$$C_{xy}(\tau) = \frac{\left\langle (x(i+\tau)-\mu_x)(y(i)-\mu_y)\right\rangle}{\sigma_x\sigma_y} \sim \tau^{-\gamma_{xy}}$$

$$F_v(s) = \frac{1}{s}\sum_{i=1}^{s}[Y_v(i)-\bar{Y}][X_v(i)-\bar{X}]$$

$$F_2^2(s) = \frac{1}{M_s}\sum_{v=1}^{M_s}\left[F_v(s)\right] \equiv \left\langle [Y_v(s)-\bar{Y}][X_v(s)-\bar{X}]\right\rangle$$

$$= sC_{xy}(0) + \sum_{i=1}^{s-1}[s-i][C_{xy}(i)+C_{xy}(-i)] \simeq s^{1-\gamma_{xy}} + s^{2-\gamma_{xy}}$$

$$F_2^2(s) \sim s^{2\lambda} \sim s^{2-\gamma_{xy}} \rightarrow 2-\gamma_{xy} = 2\lambda \rightarrow \gamma_{xy} = 2-2\lambda$$

Cross-Correlation in the presence of trends

$$x(i,j) \qquad y(i,j) \qquad i = 1,\ldots,M \qquad j = 1,\ldots,N$$

$$F(s,v) = \frac{1}{s}\sum_{i}^{s}\{Y[(v-1)s+i] - y_v(i)\} \times \{X[(v-1)s+i] - x_v(i)\}$$

$$F_2^2(s) = \left\{\frac{1}{M_s}\sum_{v=1}^{M_s}[F(s,v)]\right\} \sim s^{2\lambda}$$

$$F_2^2(s) \sim s^{2\lambda} \sim s^{2-\gamma_{xy}} \rightarrow 2-\gamma_{xy} = 2\lambda \rightarrow \gamma_{xy} = 2-2\lambda$$

$$if \quad x \equiv y \rightarrow \gamma_{xx} = 2-2H$$

Strategy for using methods

- For stationary anti-correlated signal i.e. $H<0.5$,

  SWV

- For stationary correlated signal, $H>0.5$,

  R/S

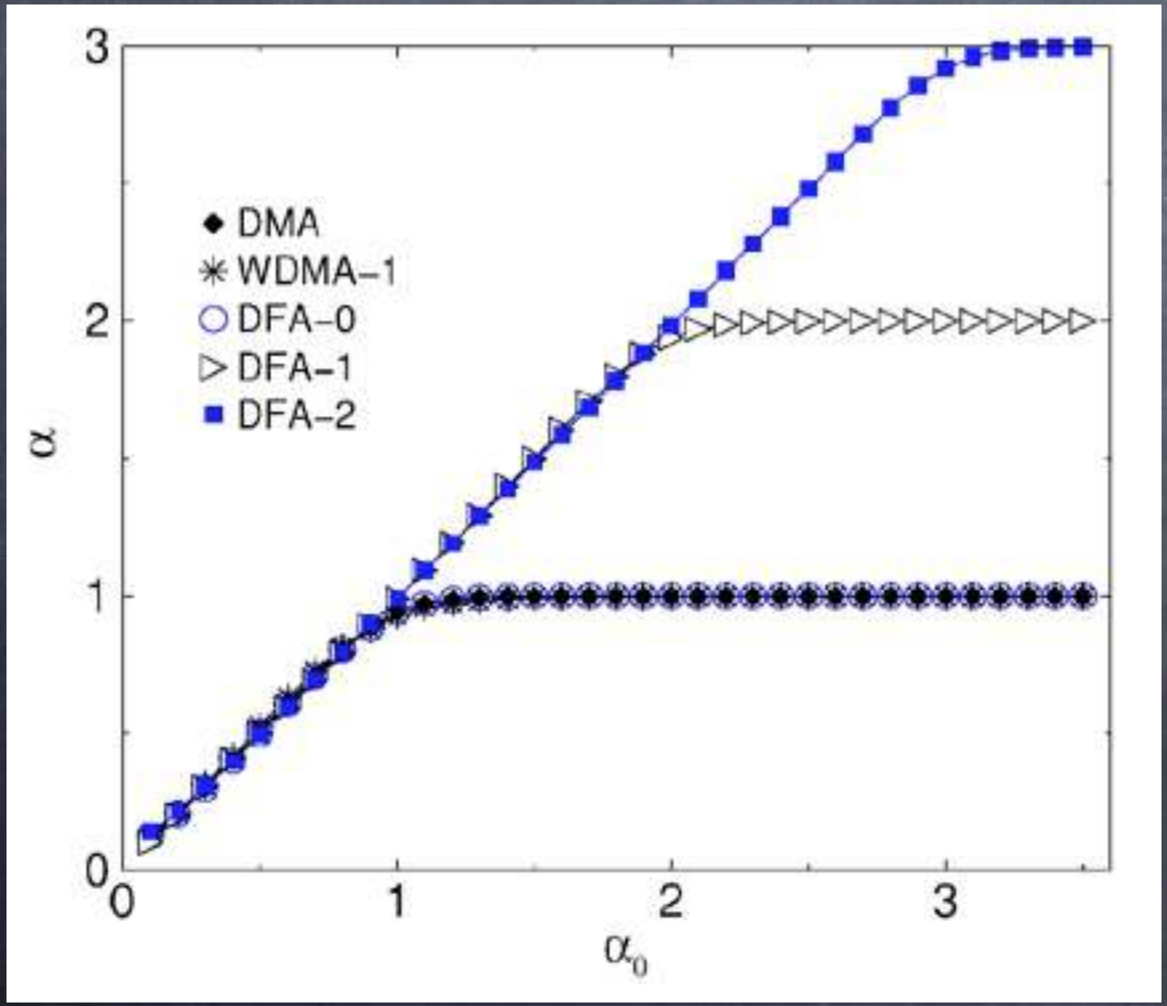
- For signal with superimposed trends,

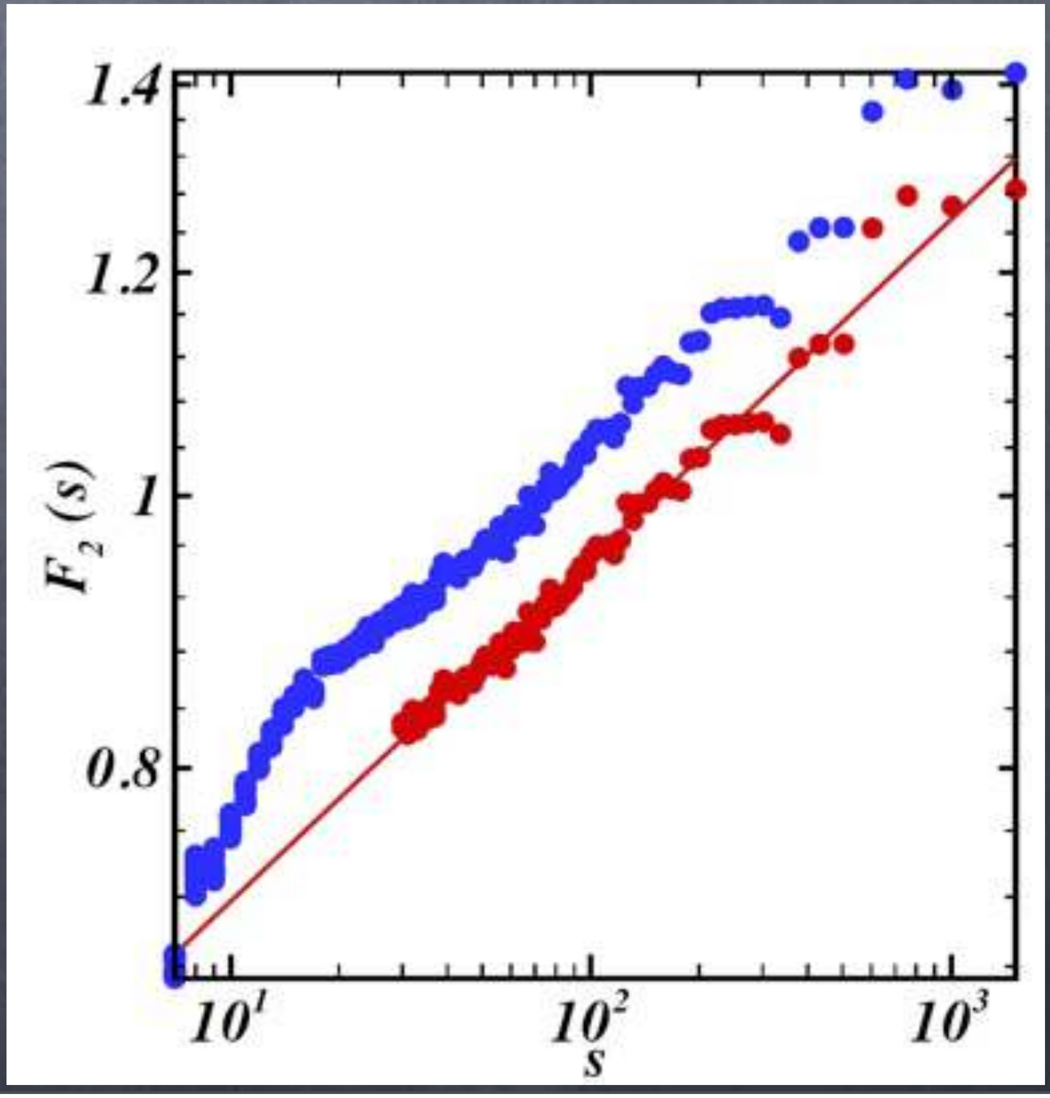
  WTMM, MF-DFA, MF-TWDFA, DMA

More about DFA

1) The longer the time series, the better the agreement with the theory in all methods but DFA behaves more reliable than others

2) DFA cannot give correct results when h(q=2)~0, In this case it is recommended to construct double profile and use DFA method

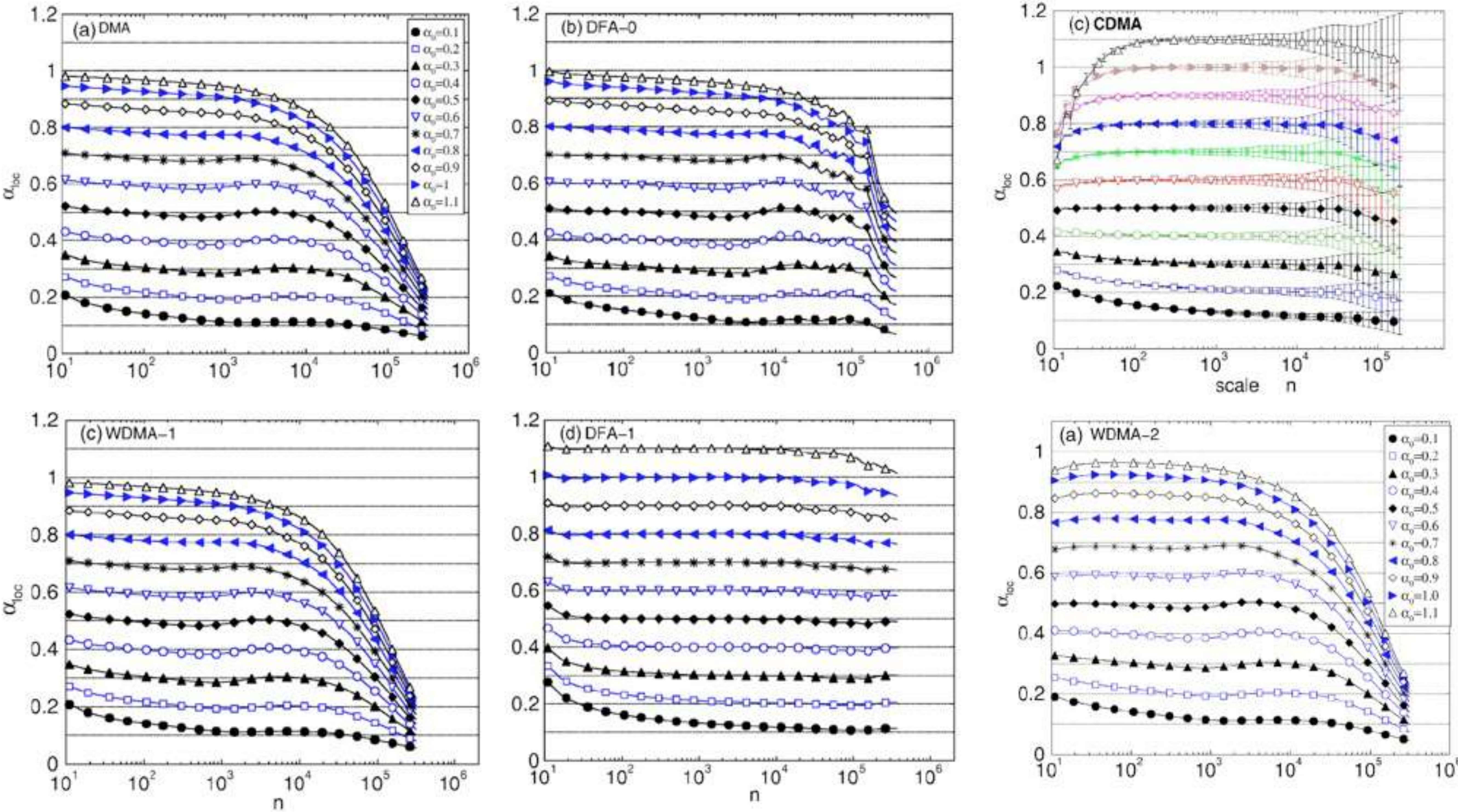




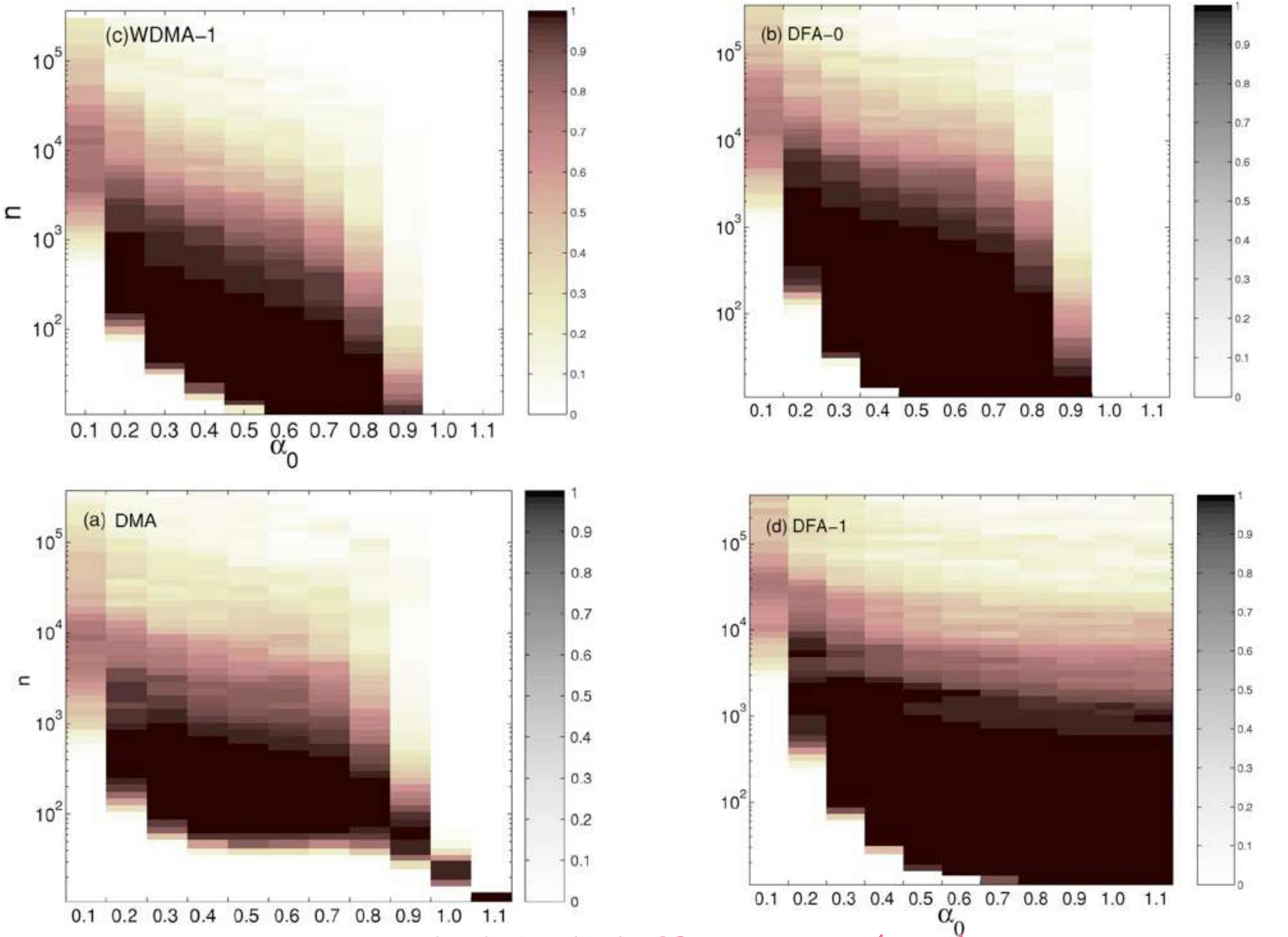
DMA (BDMA & CDMA) and MF-TWDFA

Refer to :

1) arXiv:cond-mat/0507395
2) PRE 71, 051101 (2005)
3) PRE 73, 016117 (2006)
4) JSTAT P06021 (2010)



Limei Xu et. al., PRE 71, 051101 (2005)



Limei Xu et. al., PRE 71, 051101 (2005)

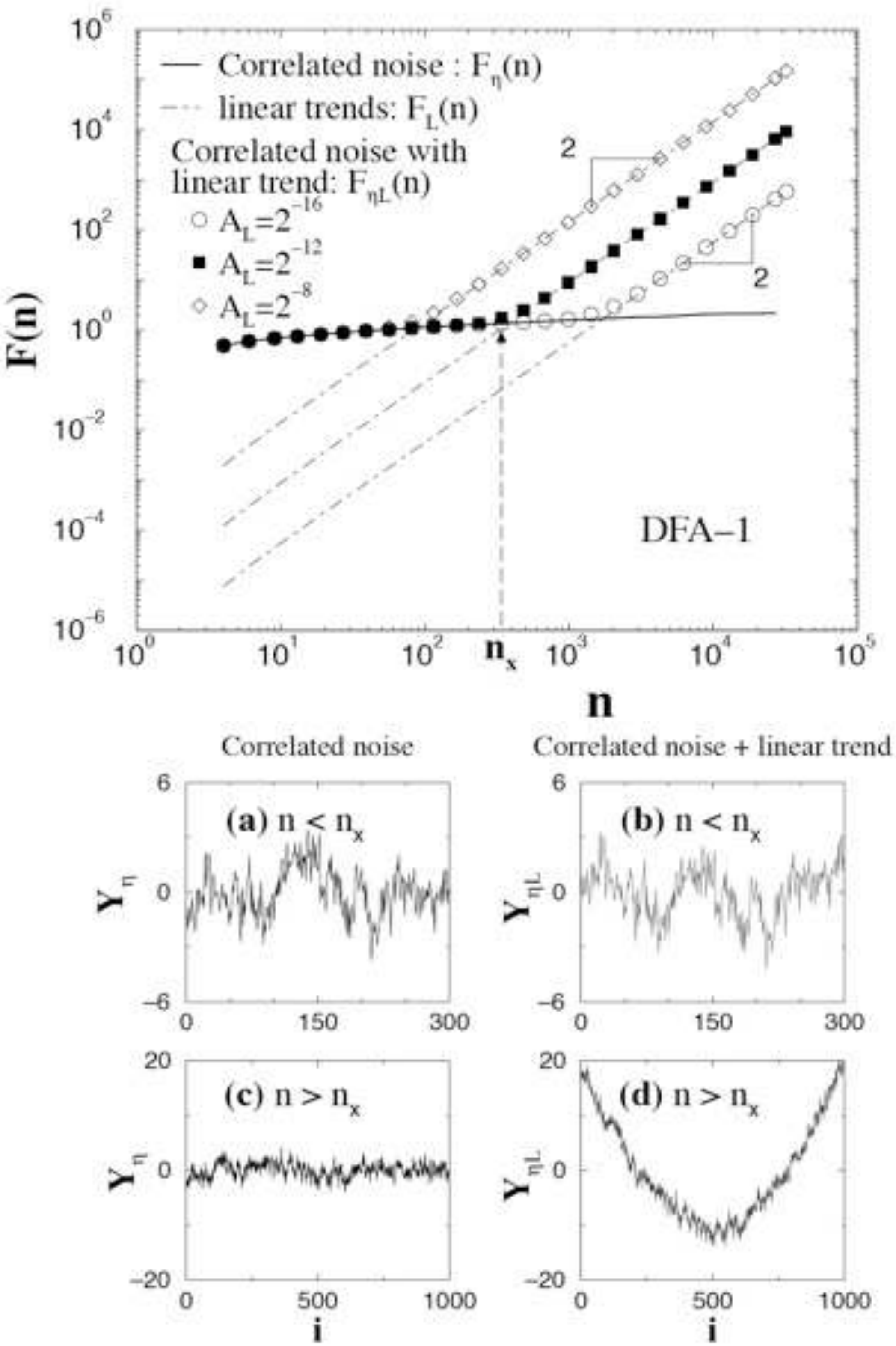
Crossover and effect of trends

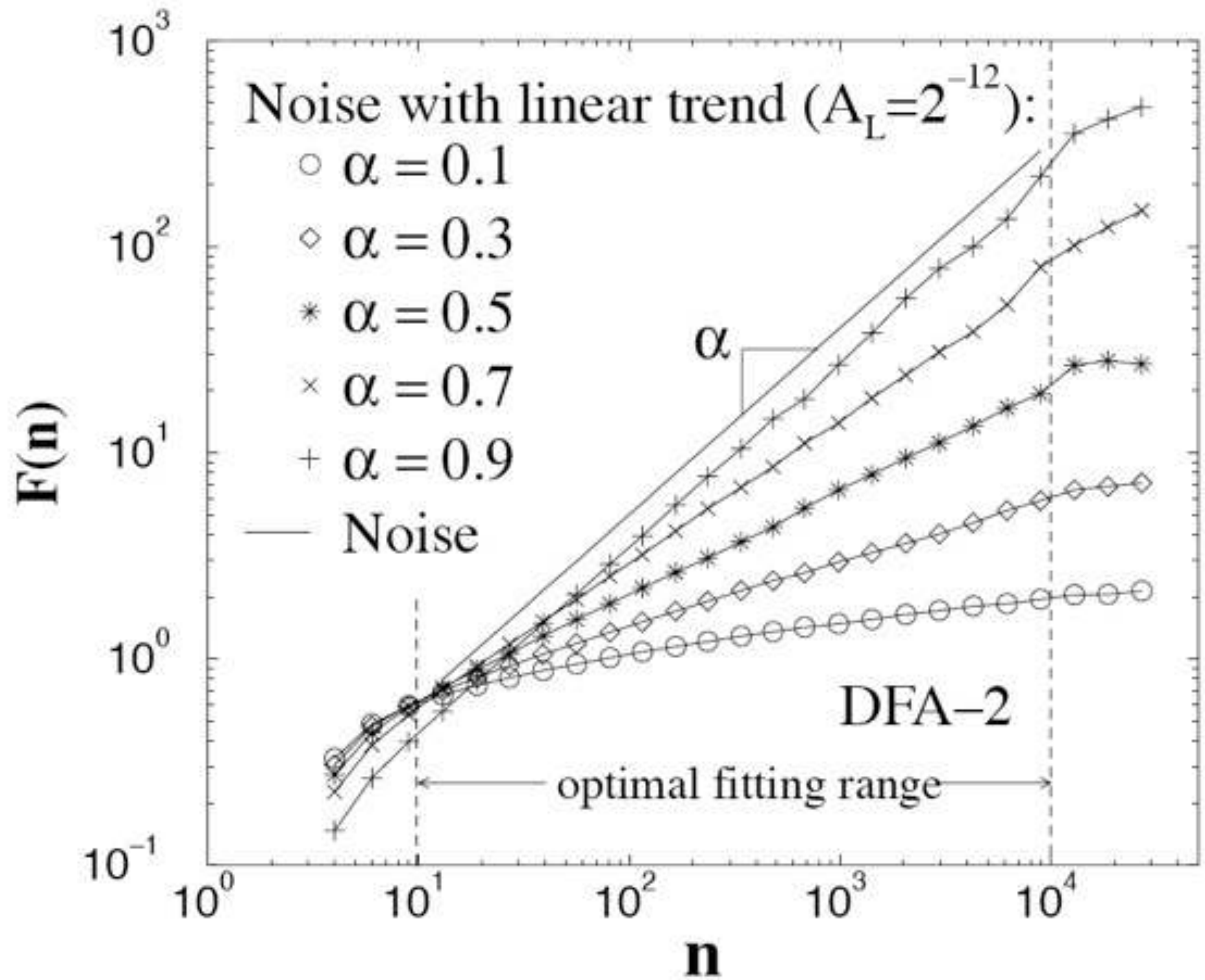
- Polynomial trends: MF-DFAm
- Sinusoidal trends: F-DFA, SVD, chaotic SVD and Empirical mode decomposition(EMD)

Z. Wu et al., PNAS, 104, 38 (2007)

Polynomial Trends

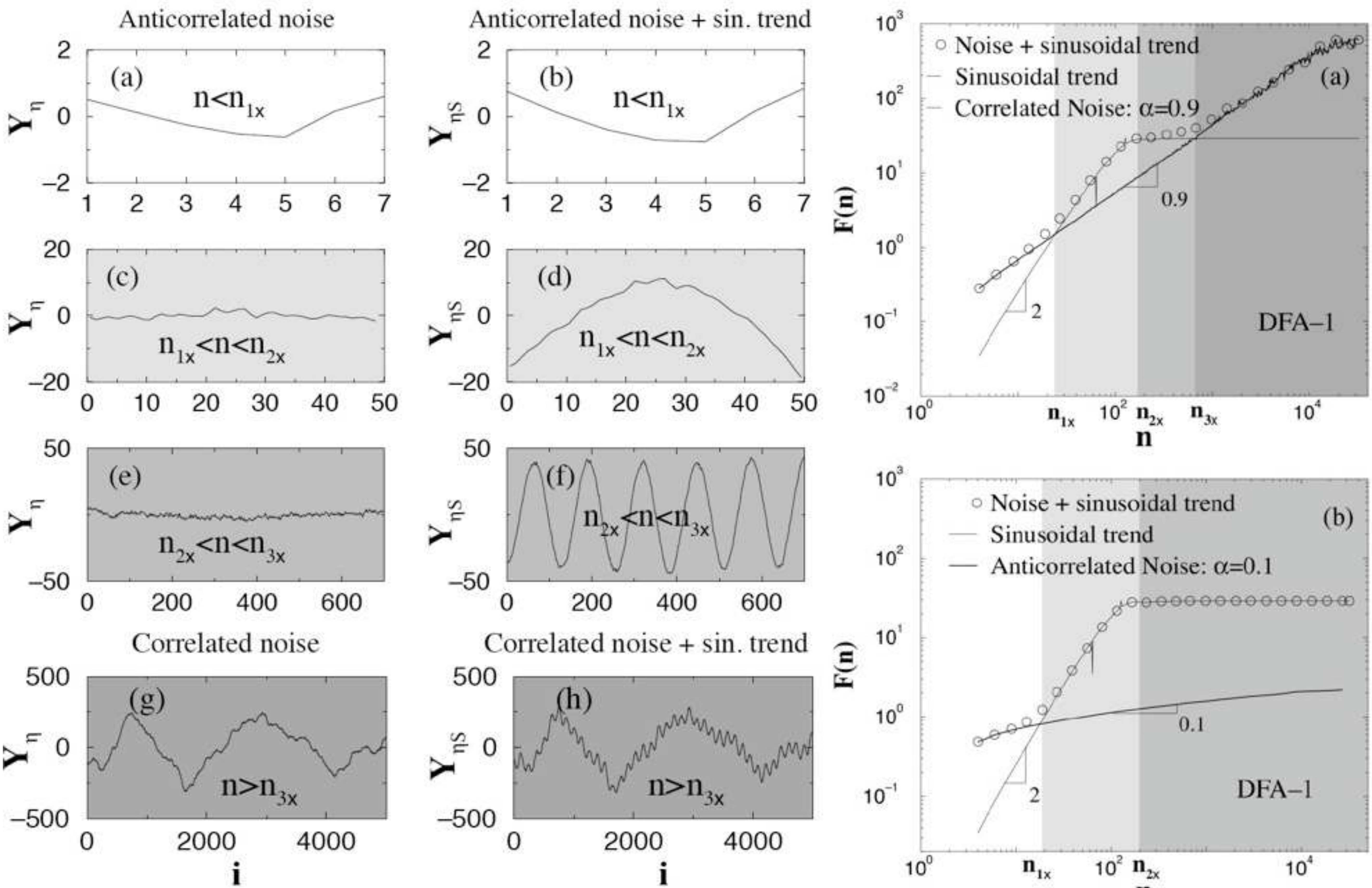
- It has been demonstrated that by MF-DFAm, polynomial of order m-1 to be diminished





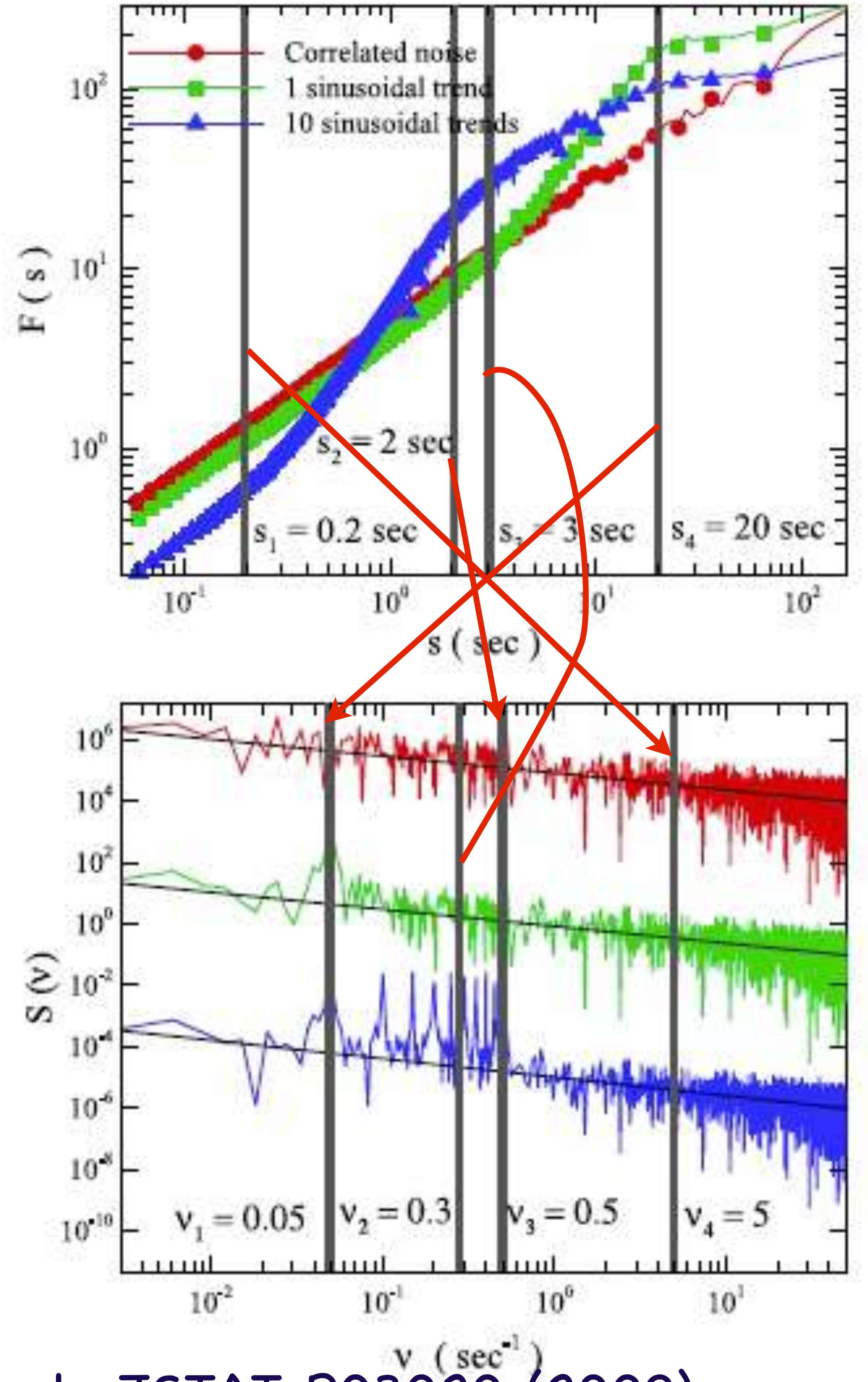
K. Hu et. al., PRE 64, 011114 (2001)

Sinusoidal Trends



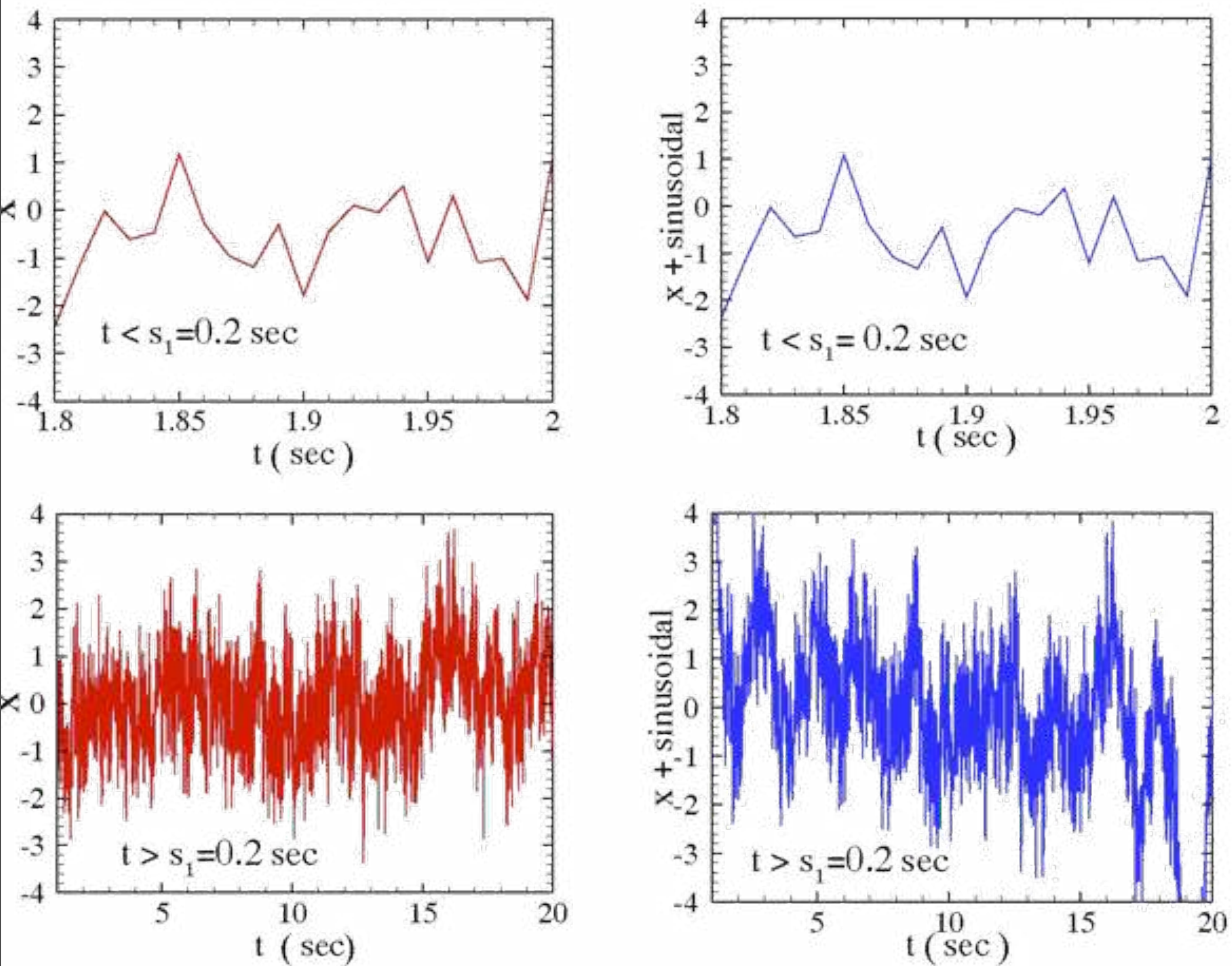
K. Hu et. al., PRE 64, 011114 (2001)

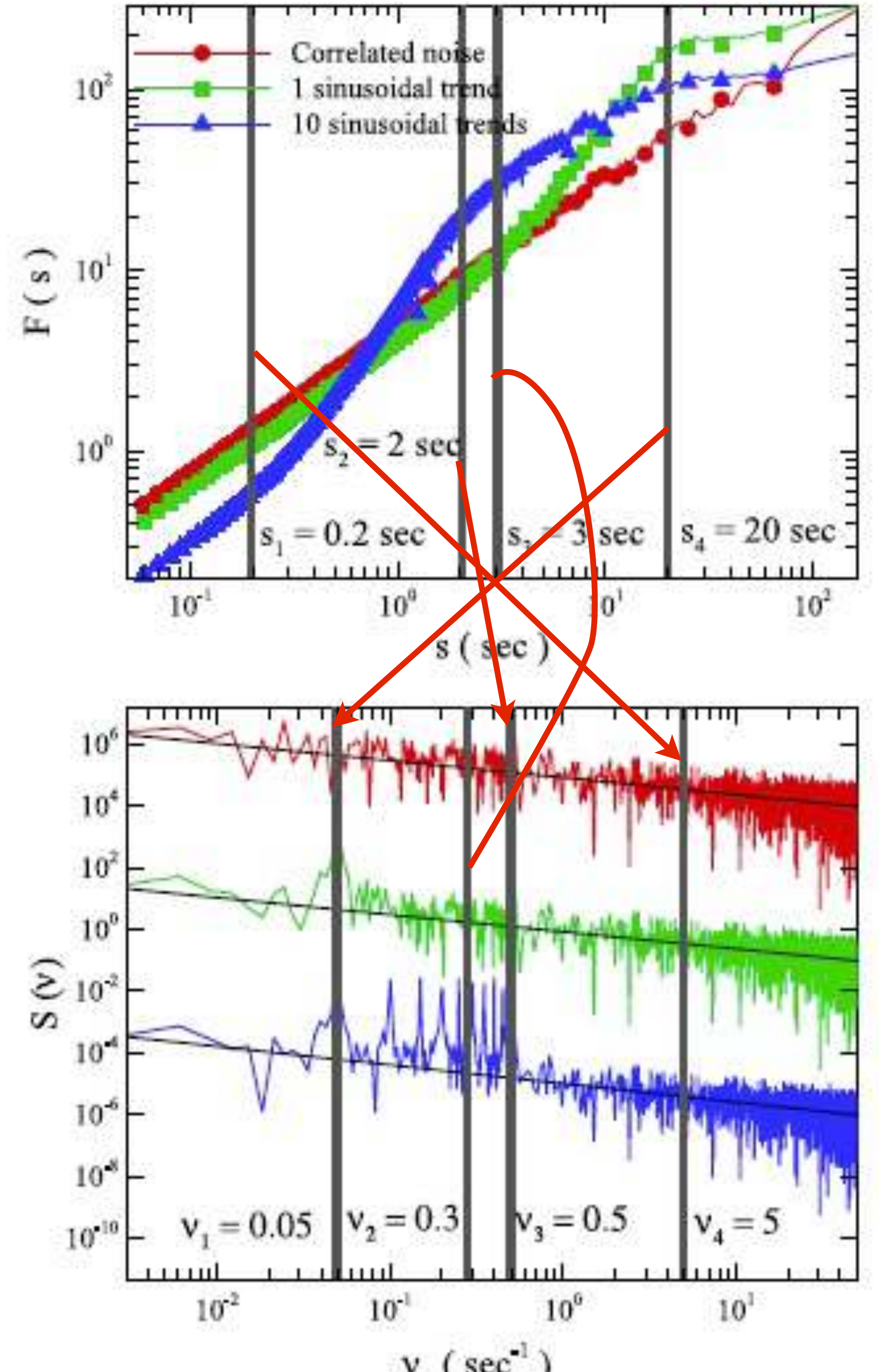
Competition between noise and sinusoidal trends



S. Kimiagar, M. Sadegh Movahed et. al., JSTAT P03020 (2009)

Competition between noise and sinusoidal trends



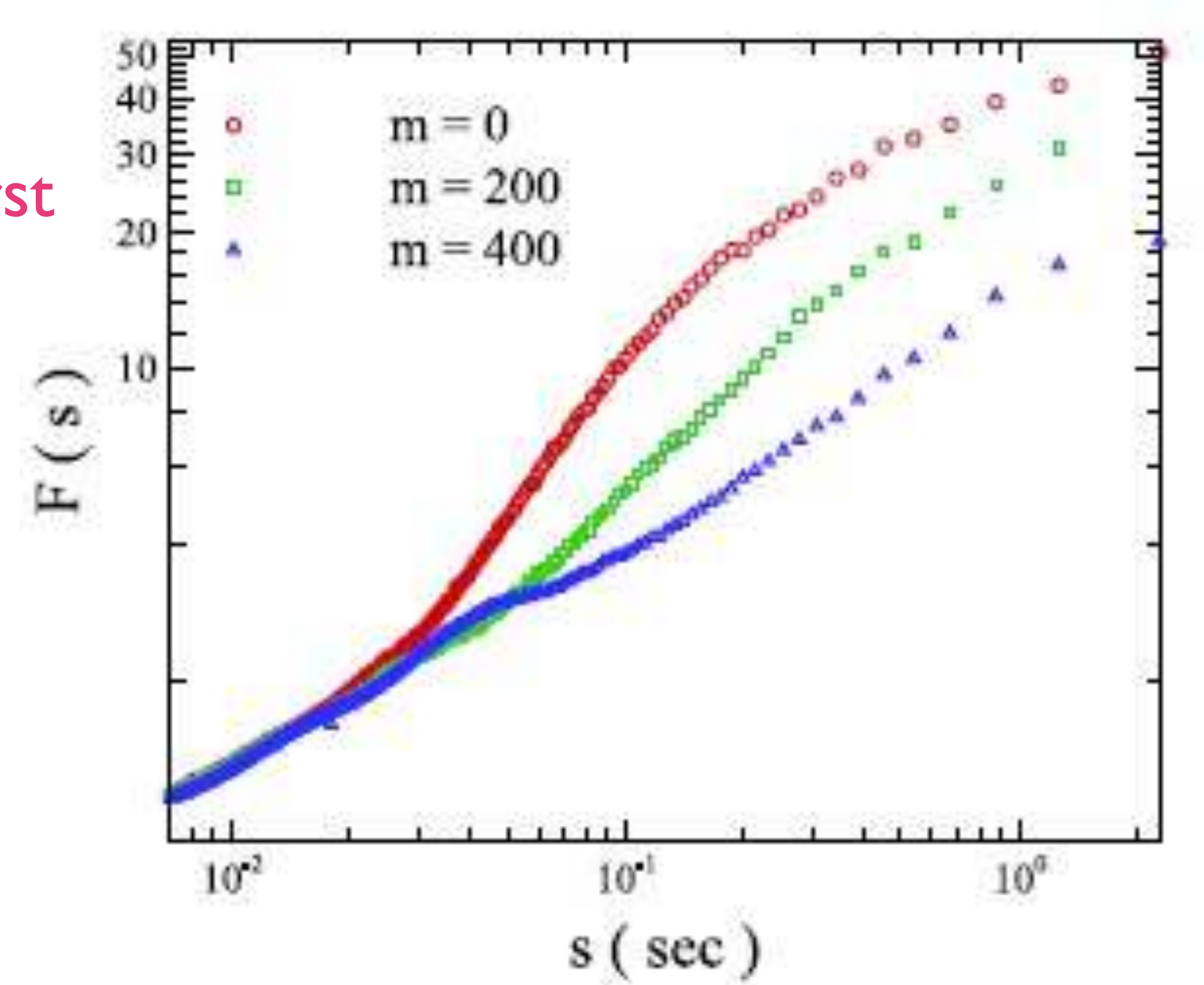


S. Kimiagar, M. Sadegh Movahed et. al., JSTAT P03020 (2009)

Fourier-Detrended

- Indeed, this method bases on High-pass filter

We transform the data set to the Fourier space and then truncate the first few coefficients of the Fourier expansion, finally by inverse transformation, the clean data will be retrieved



Physica A 357, 447-454 (2005); Physica A 354, 182-198 (2005); Chaos, Solitons and fractals 26, 777-784 (2005), Jstat P03020 (2009)

Singular Value Decomposition (SVD)

$$\{x_i\}; i = 1, ..., N \qquad d \leq N - (d-1)\tau + 1$$

$$\mathbf{\Gamma} \equiv \begin{pmatrix} x_1 & x_{1+\tau} & \cdots & x_{1+N-(d-1)\tau-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_i & x_{i+\tau} & \cdots & x_{i+N-(d-1)\tau-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_d & x_{d+\tau} & \cdots & x_{d+N-(d-1)\tau-1} \end{pmatrix}$$

p will be given by power spectrum

$$\mathbf{\Gamma} = \mathbf{USV}^\dagger$$

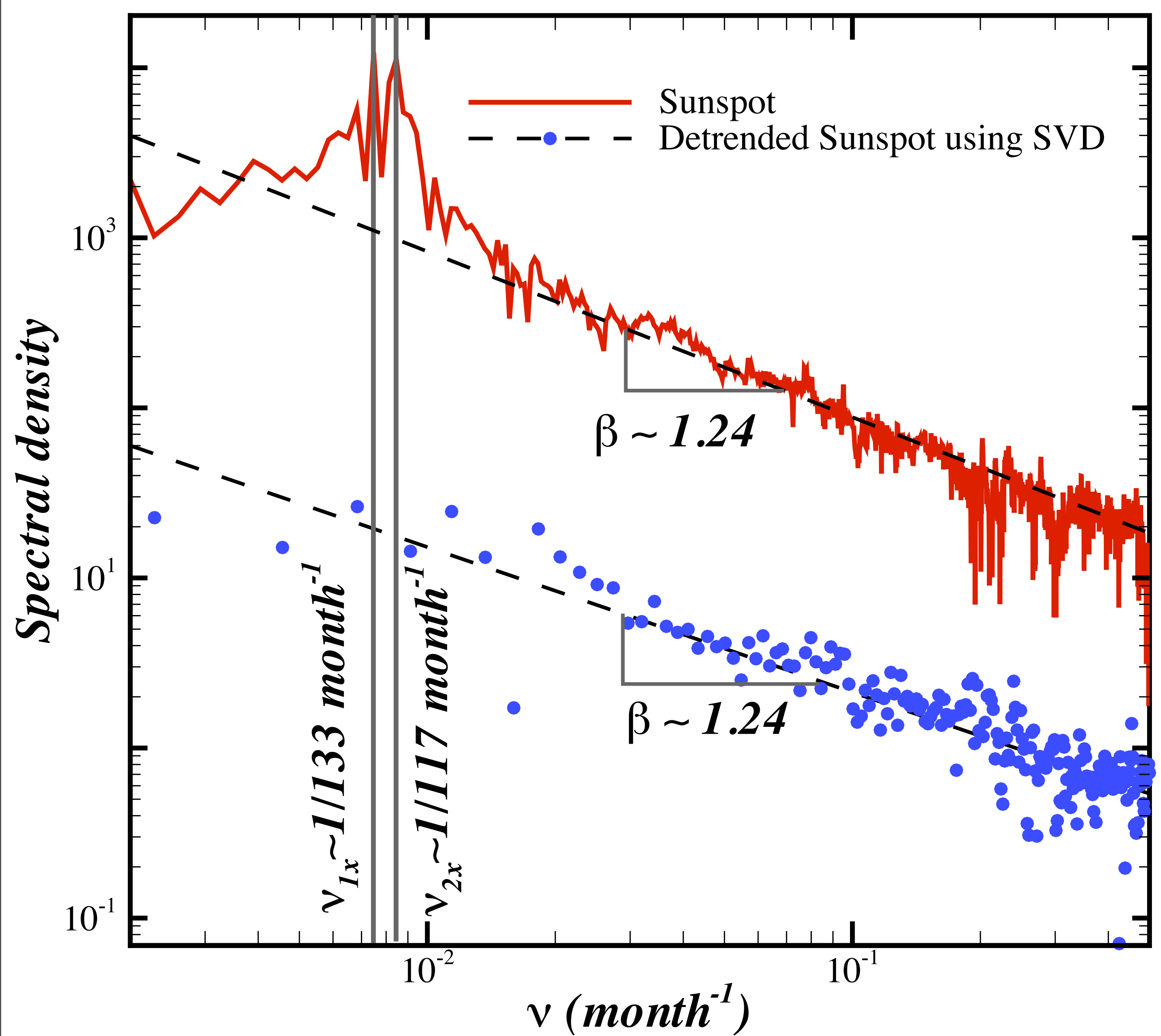
$$\mathbf{\Gamma}^\dagger\mathbf{\Gamma}\mathbf{v_i} = \lambda_i^2\mathbf{v_i}$$

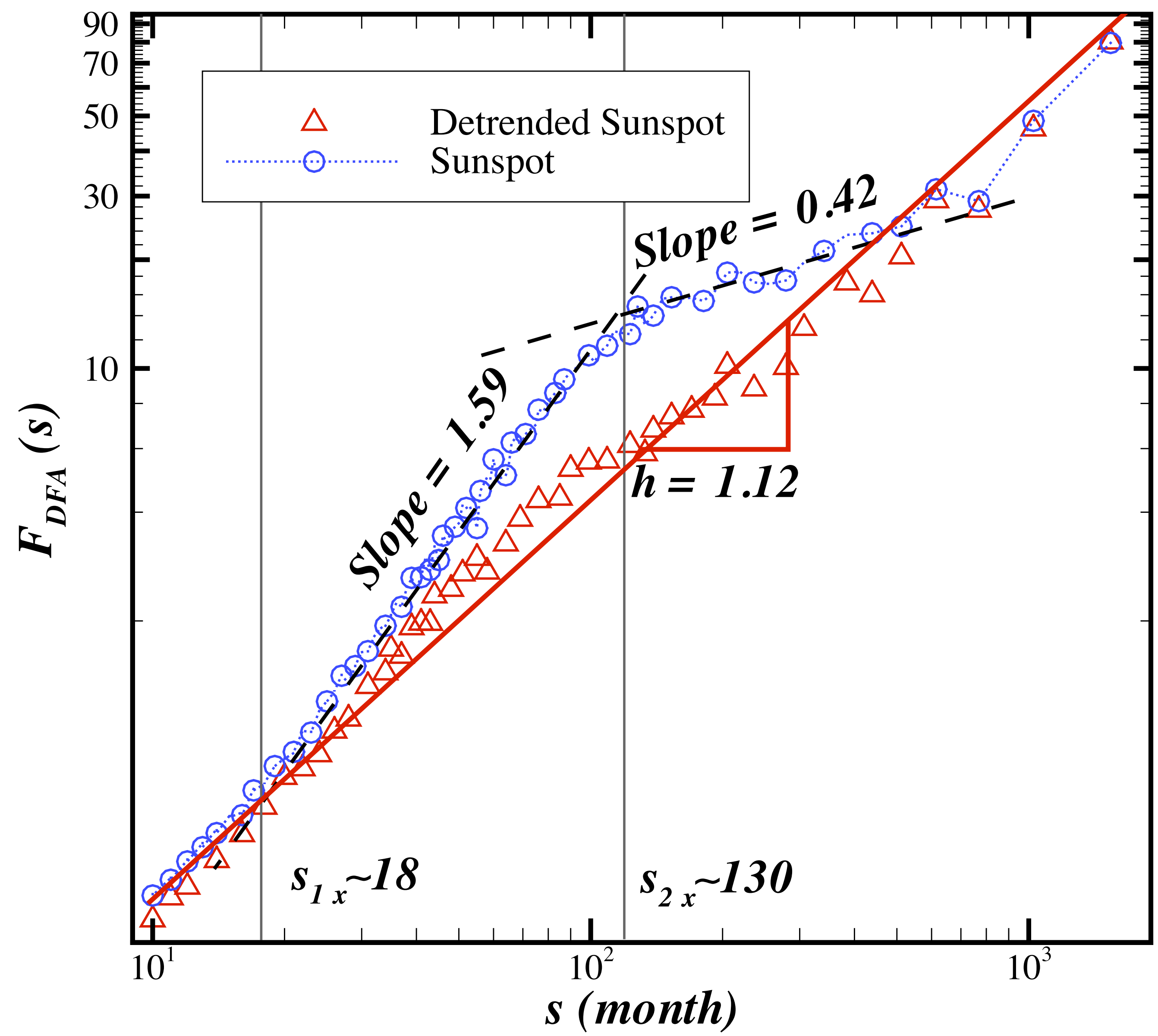
$$\mathbf{\Gamma}\mathbf{\Gamma}^\dagger\mathbf{u_i} = \lambda_i^2\mathbf{u_i}$$

$$x^*_{i+j-1} = \Gamma^*_{ij}$$

The p dominant eigenvalue and associating eigendecomposed vector represent the superimposed trend and the remaining (d-p) demonstrates intrinsic fluctuations

S. hajian and M. Sadegh Movahed, arXiv:0908.0132



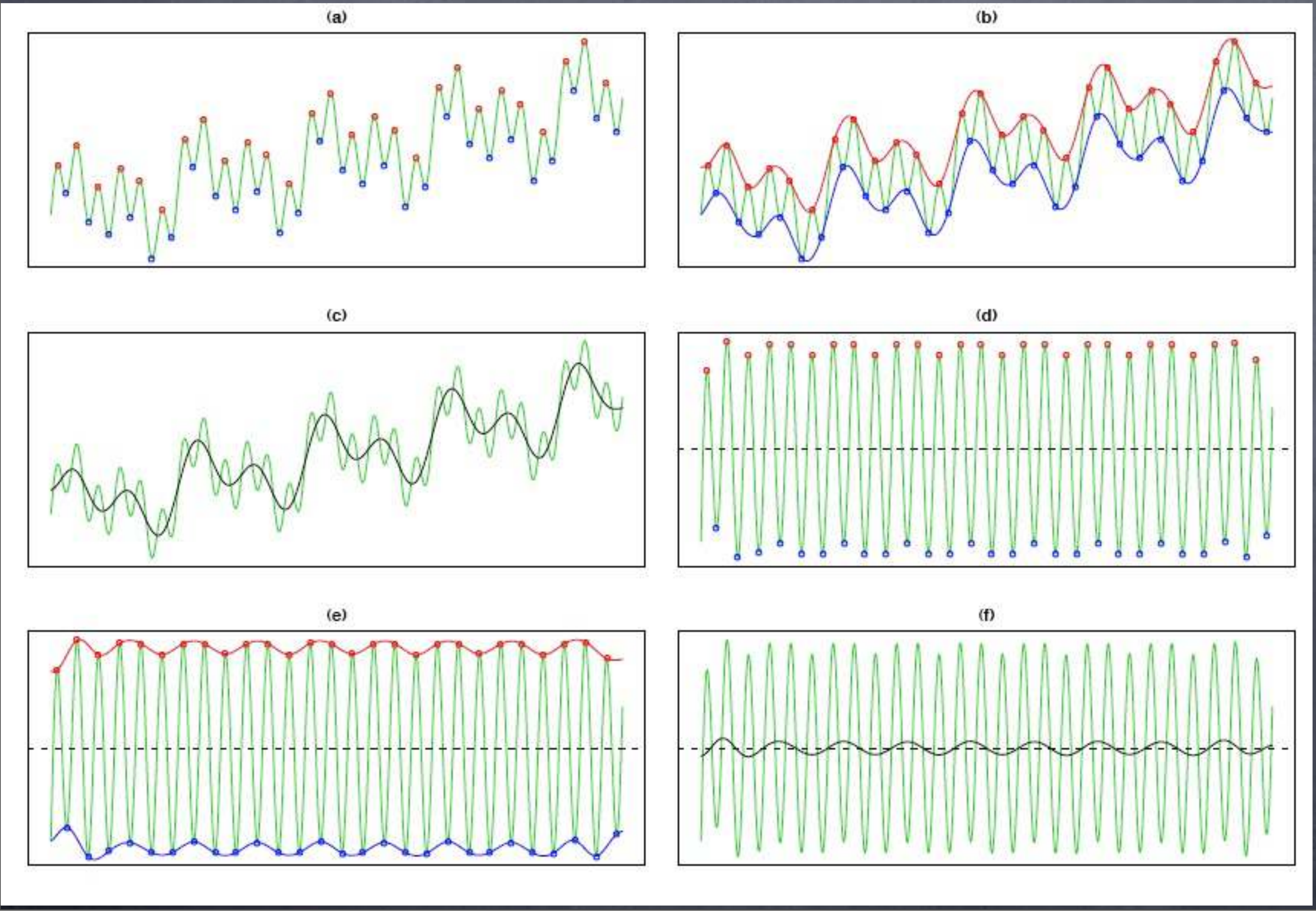


S. hajian and M. Sadegh Movahed, arXiv:0908.0132

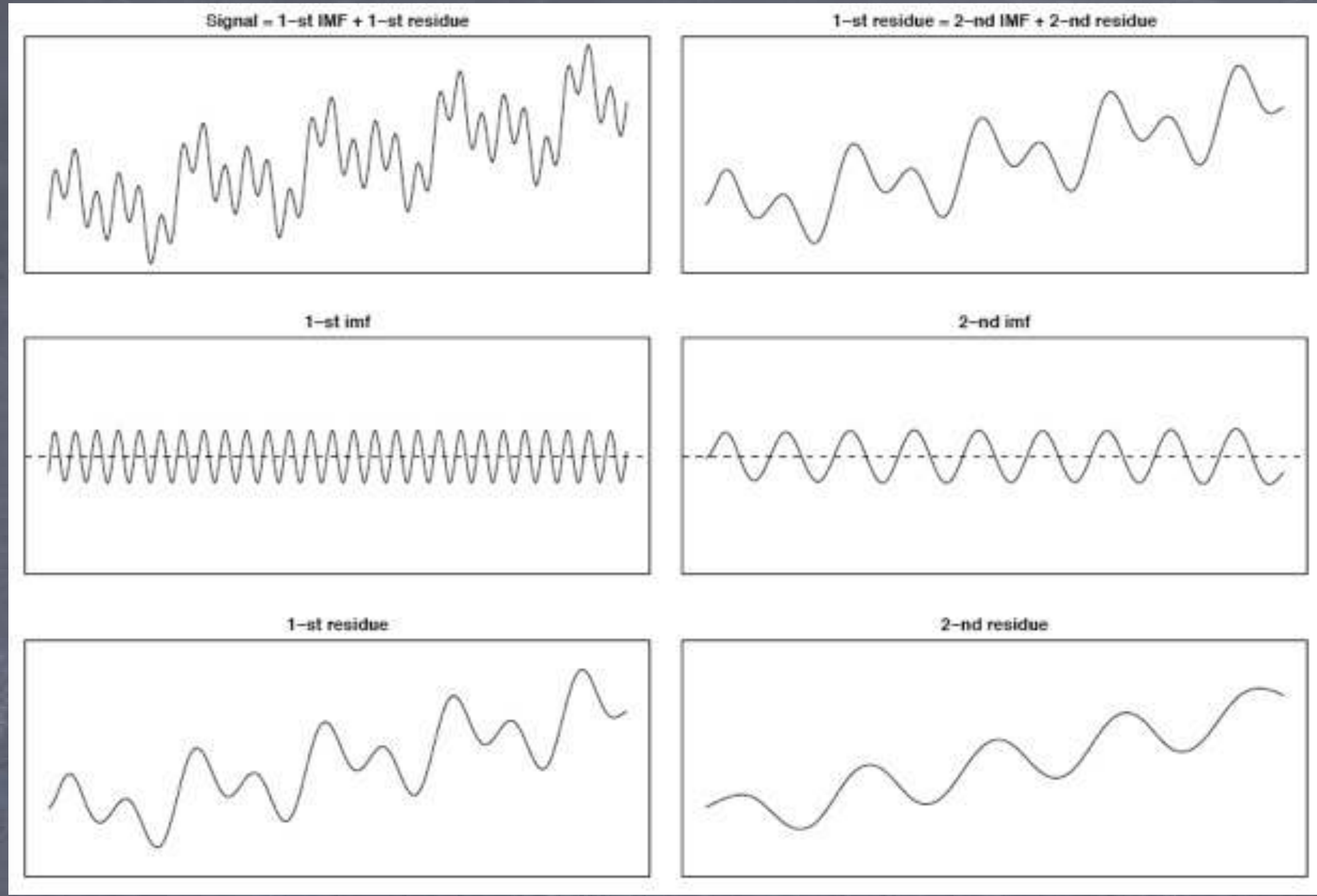
Empirical Mode decomposition (EMD)

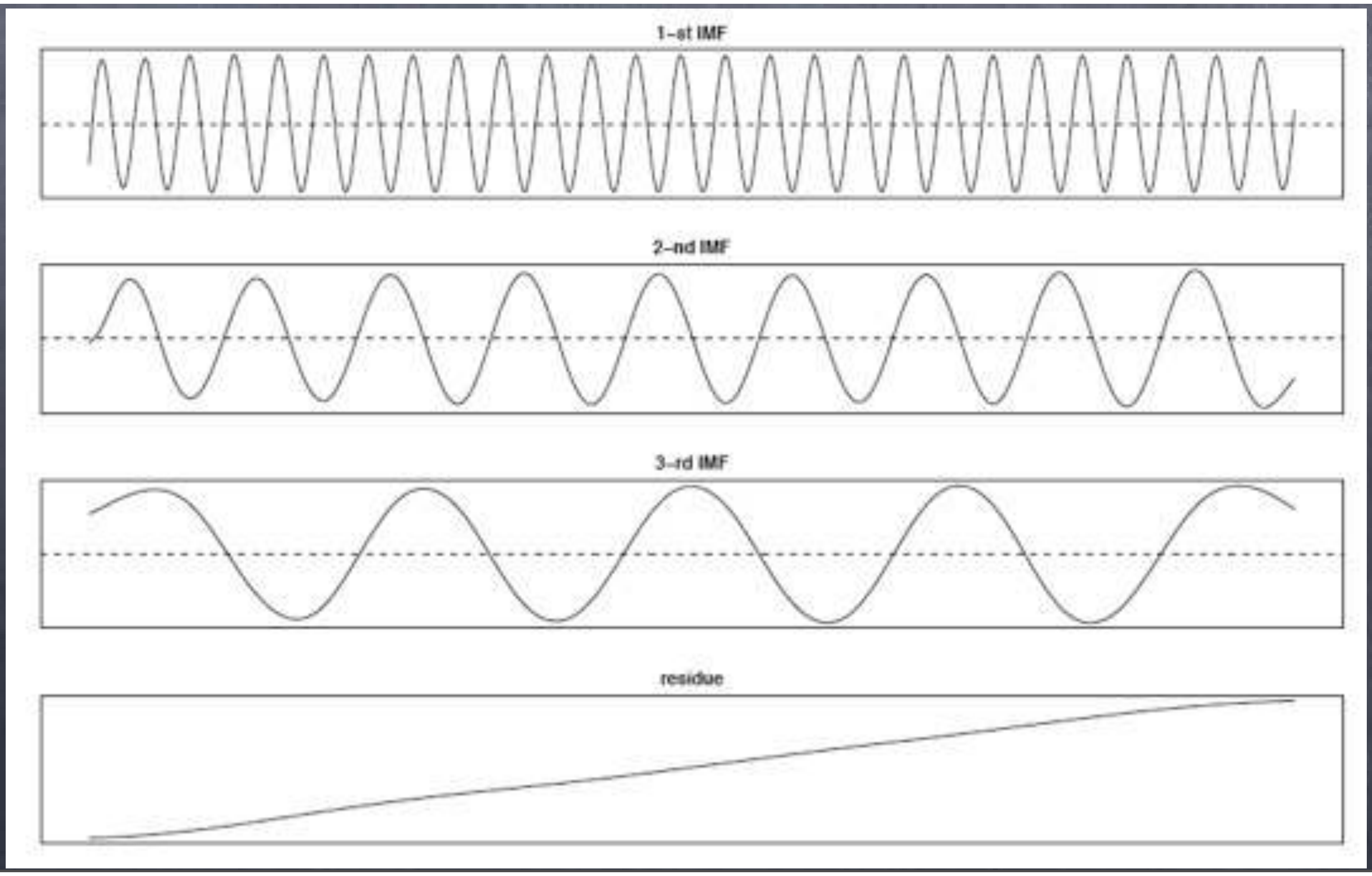
- This method is known as non-parametric method
- There is a good review by Norden E. Huang
  Proceedings: Mathematical, Physical and Engineering Sciences, Vol. 454, No. 1971 (Mar. 8, 1998)
- In this case, the intrinsic mode functions (IMFs) satisfy two conditions:
  1) The number of extrema and zero-crossing differs only by one
  2) The local average is zero

1) Identify the local extrema and find their average (Generating upper envelop and lower envelope)
2) Subtracting the envelop mean from signal
3) Check the IMF conditions

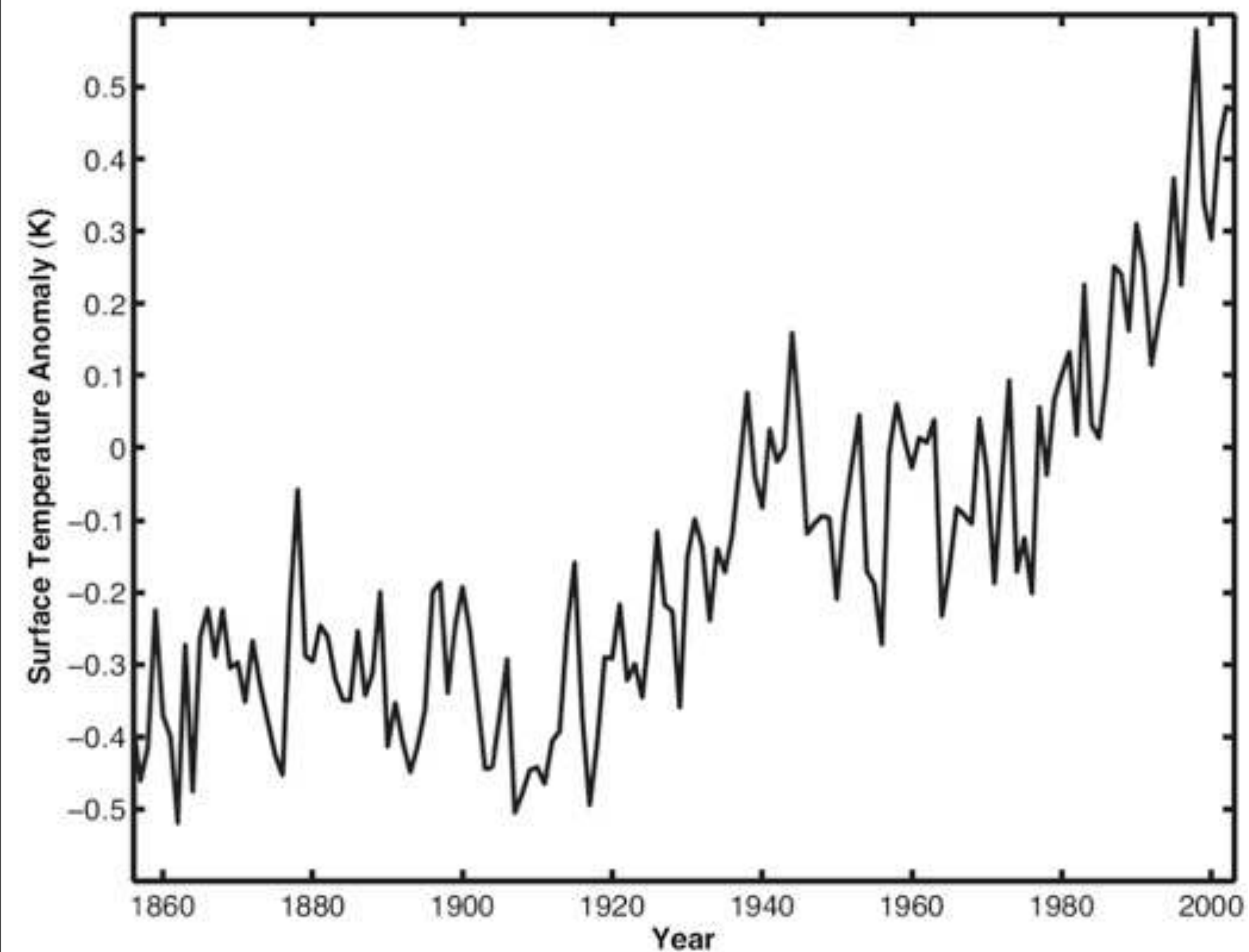


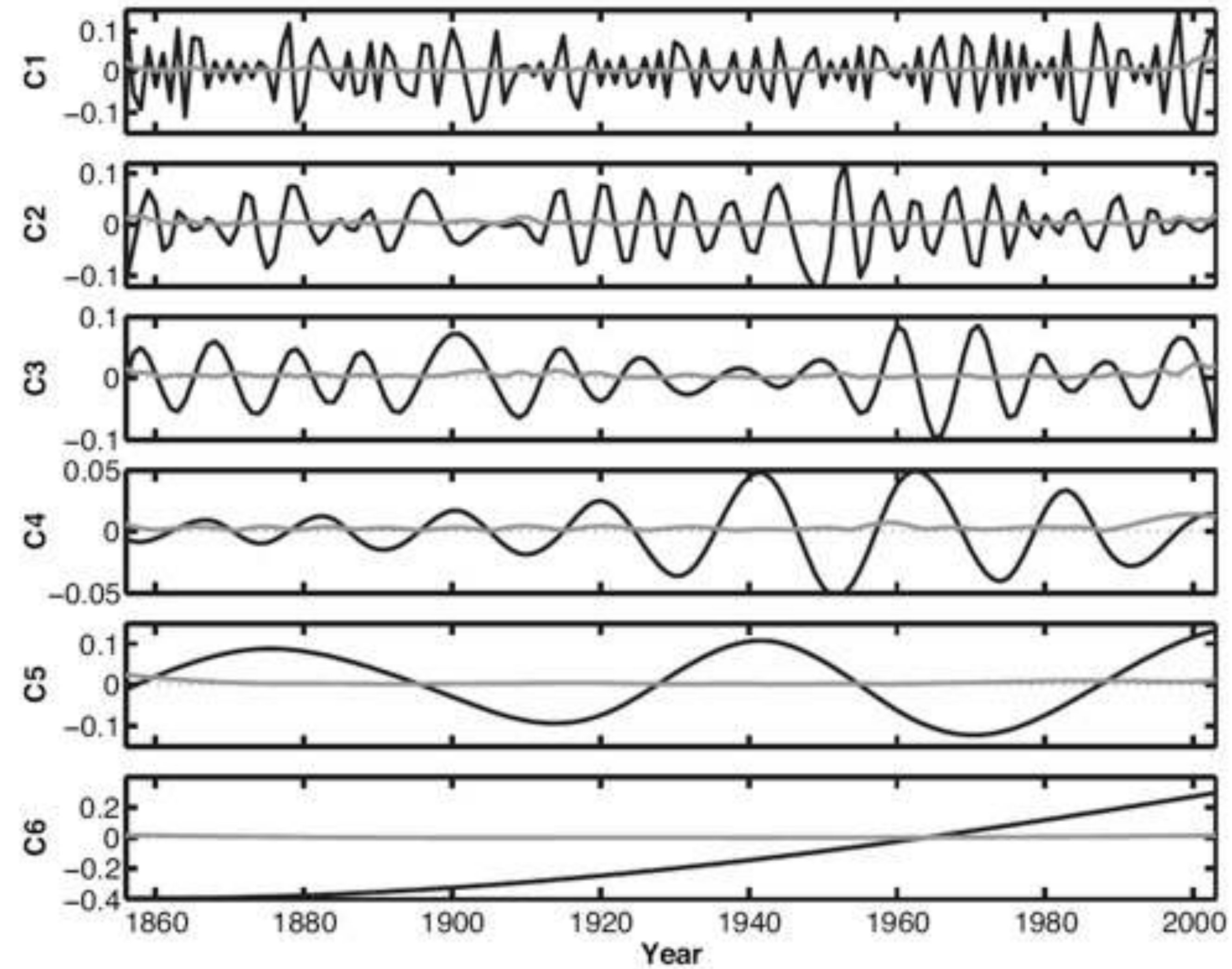
D. Kim et. all.,R Journal, Vol.1, 1 may 2009

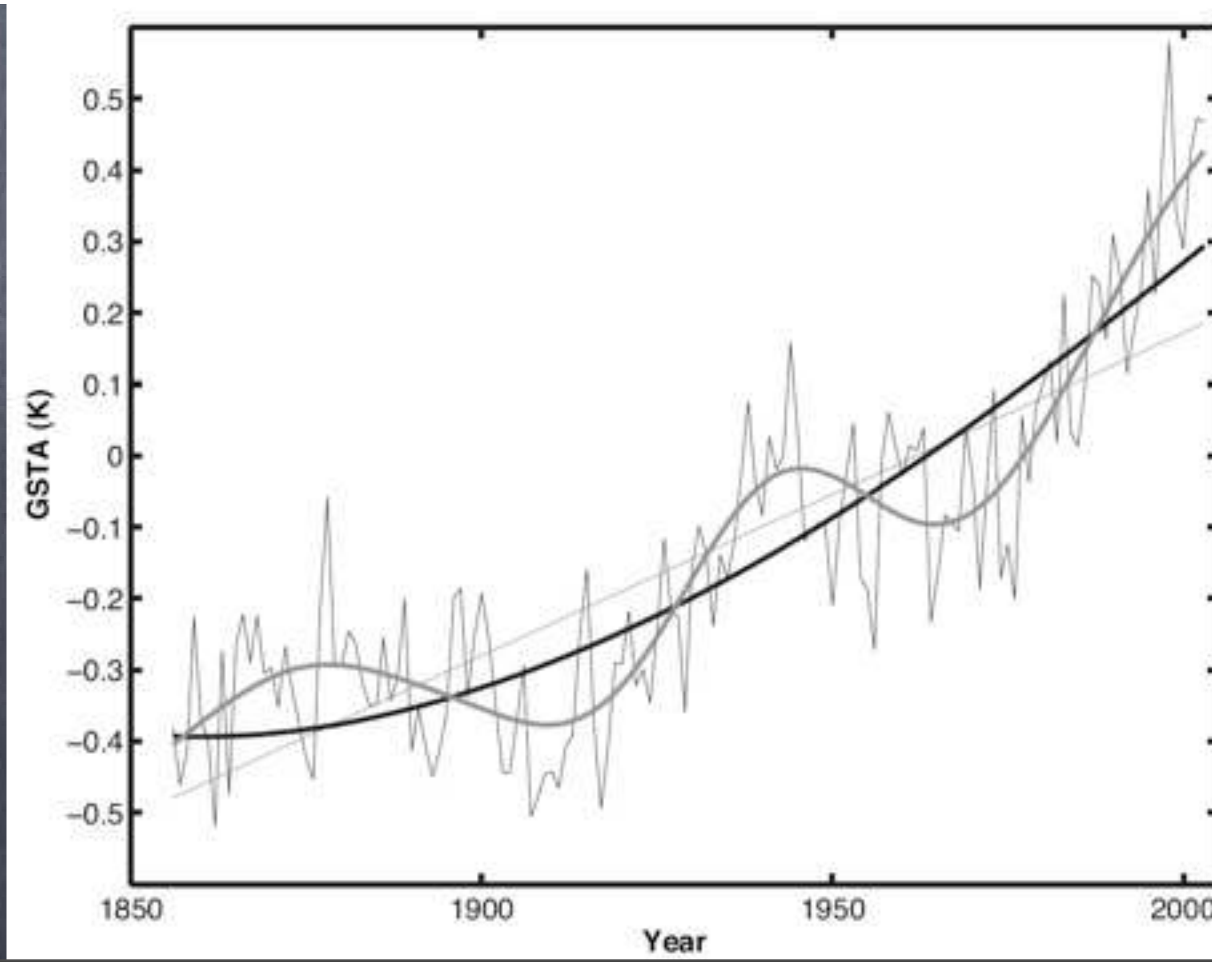




D. Kim et. all.,R Journal, Vol.1, 1 may 2009







gray corresponds t
variance and black
corresponds to IMF
various stopping cri

Z. Hu, et. al., PNAS, 104, 38 (2007)

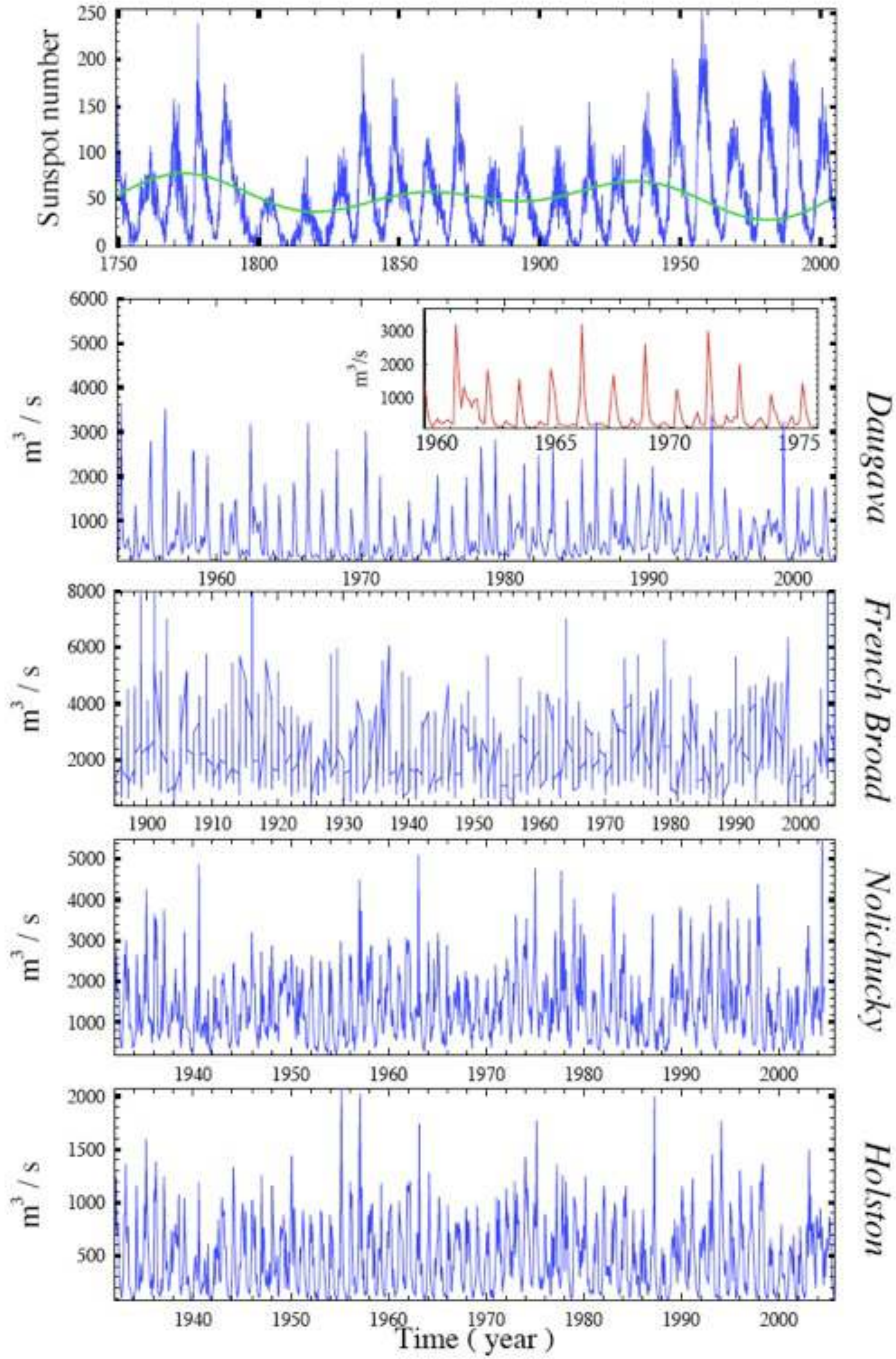
Advantages and disadvantages

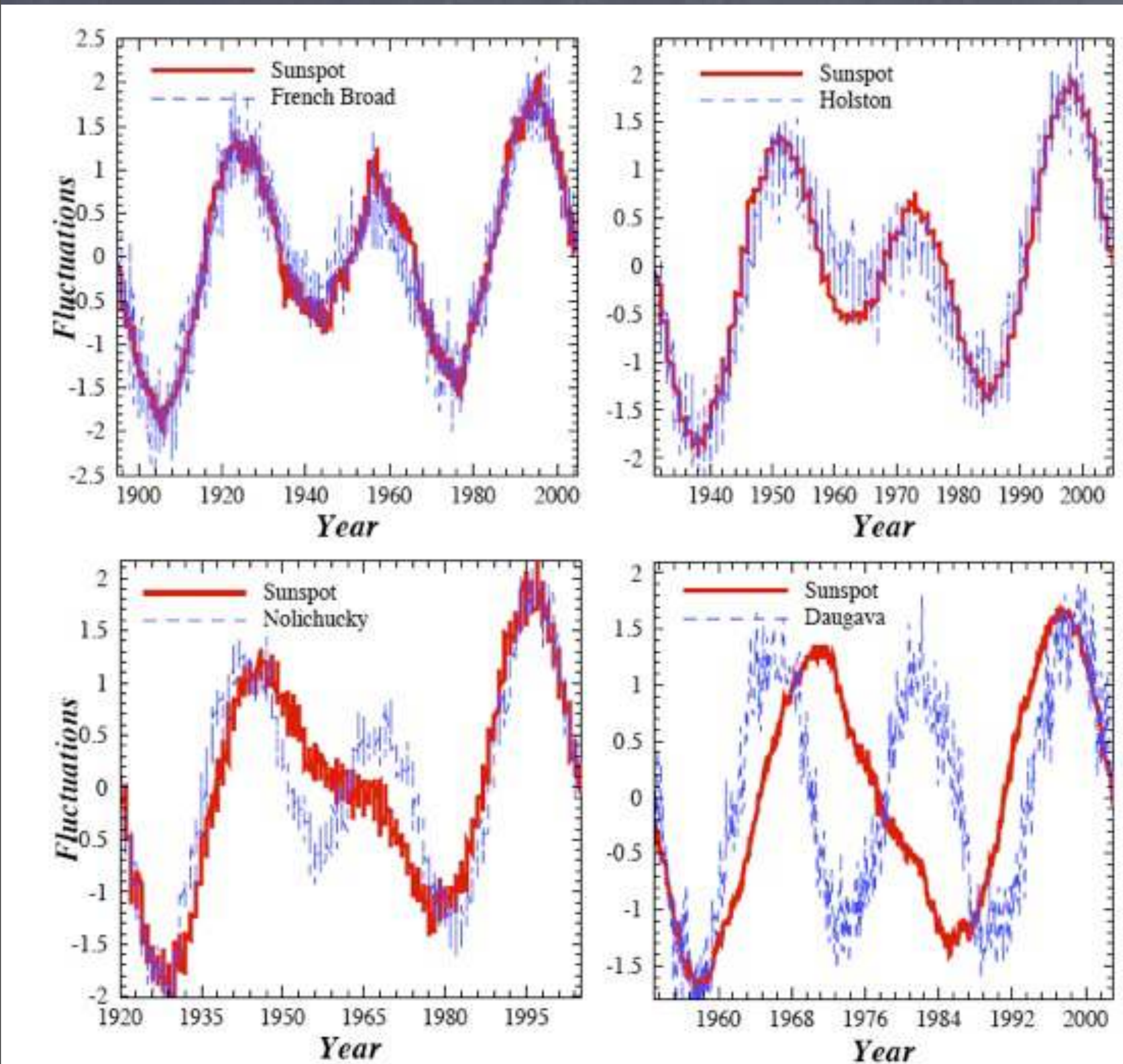
- The size of underlying data won't be invariant by using F-DFA, while the size will be preserved in SVD and EMD

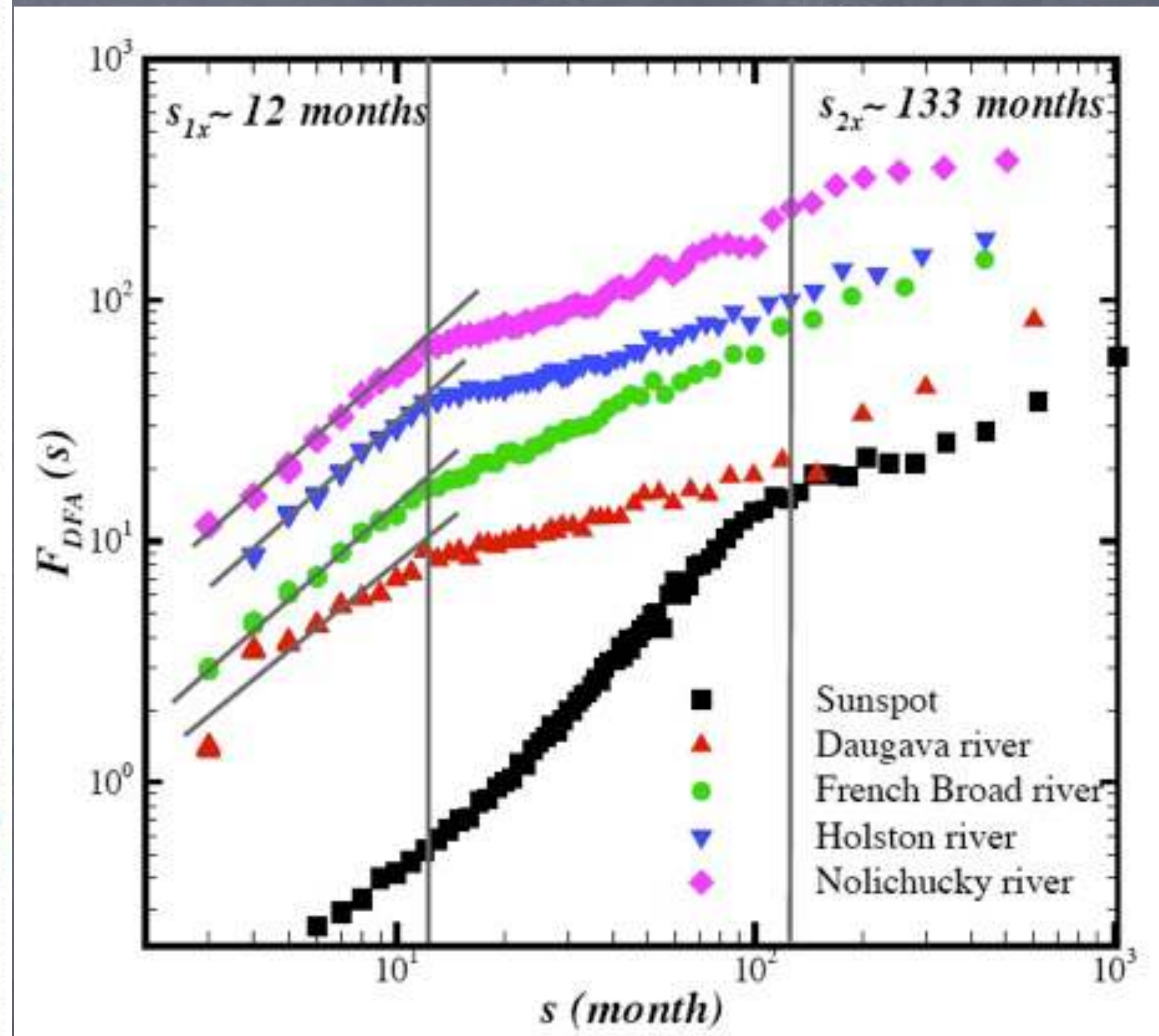
As an example: Application of DCCA Sunspot and River flow

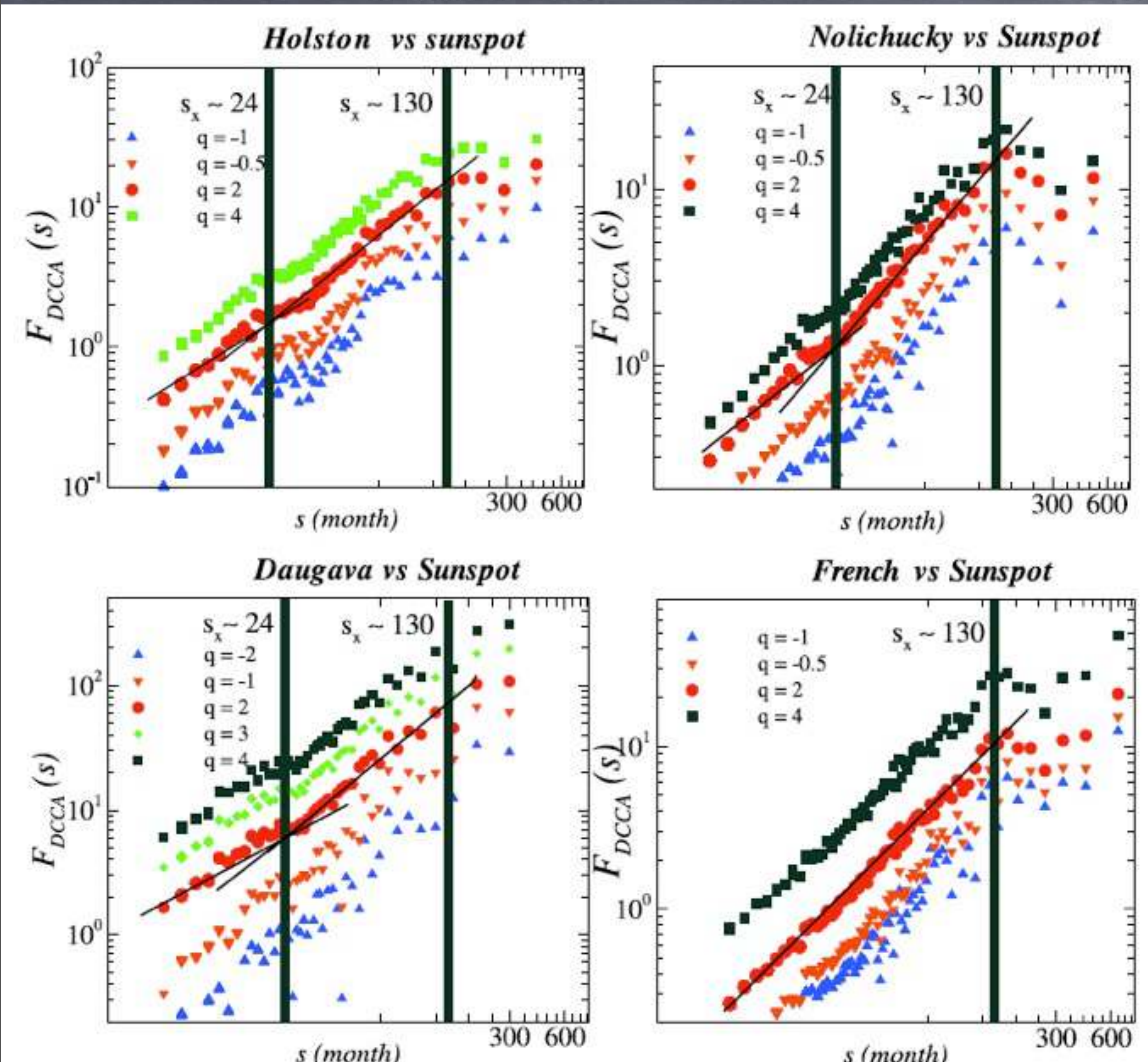
solar irradiance to be increases when sun activity increases and the cloud cover to be increase resulting stream flow increase

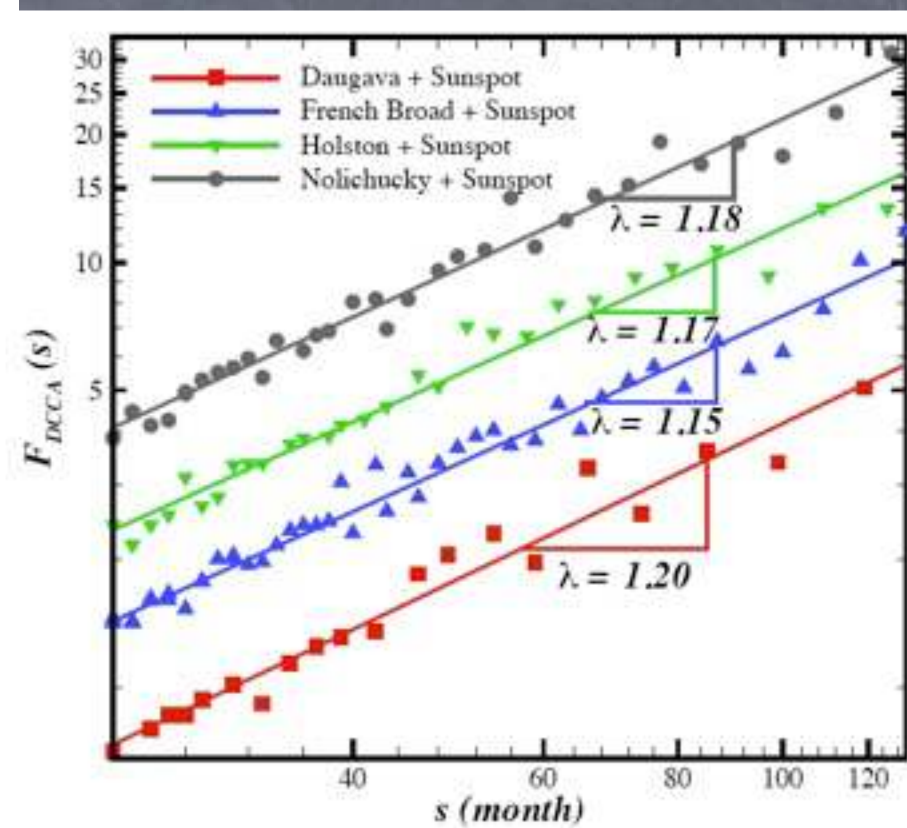
River	Discharge (m^3/s)	Series length	Drainage area (km^2)	Location
French broad	2093	1896.1–2005.12	2448	35°36′ N 82°34′ W
Daugava	601	1953.1–2002.12	87,900	56°57′ N 24°6′ E
Holston	479	1931.1–2005.12	784.8	36°39′ N 81°50′ W
Nolichucky	1379	1921.1–2005.12	2085	36°10′ N 82°27′ W


Sunspot number
m^3 / s
Daugava
French Broad
Nolichucky
Holston
Time (year)

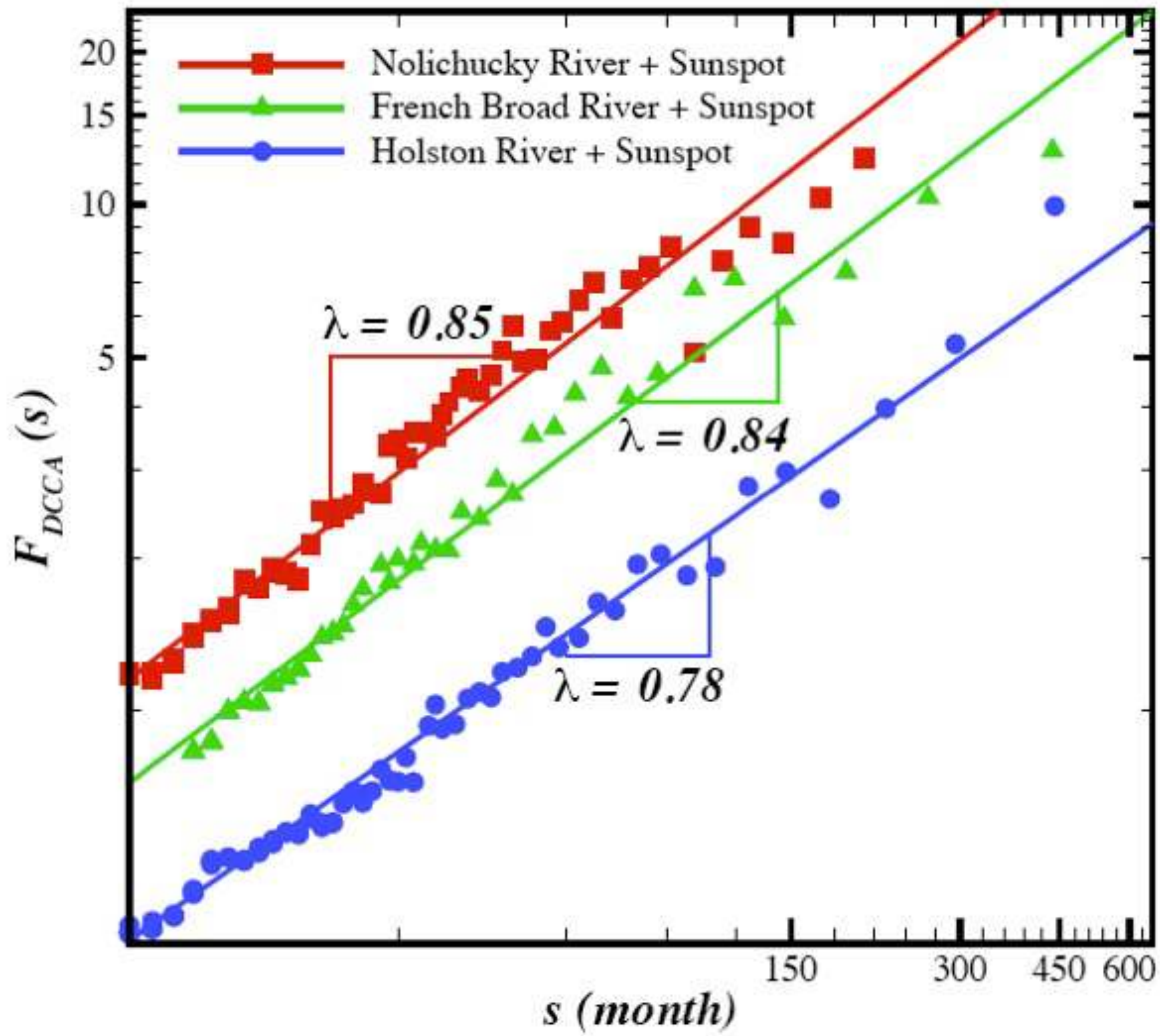

Sunspot
French Broad
Sunspot
Holston
Sunspot
Nolichucky
Sunspot
Daugava
Fluctuations
Year

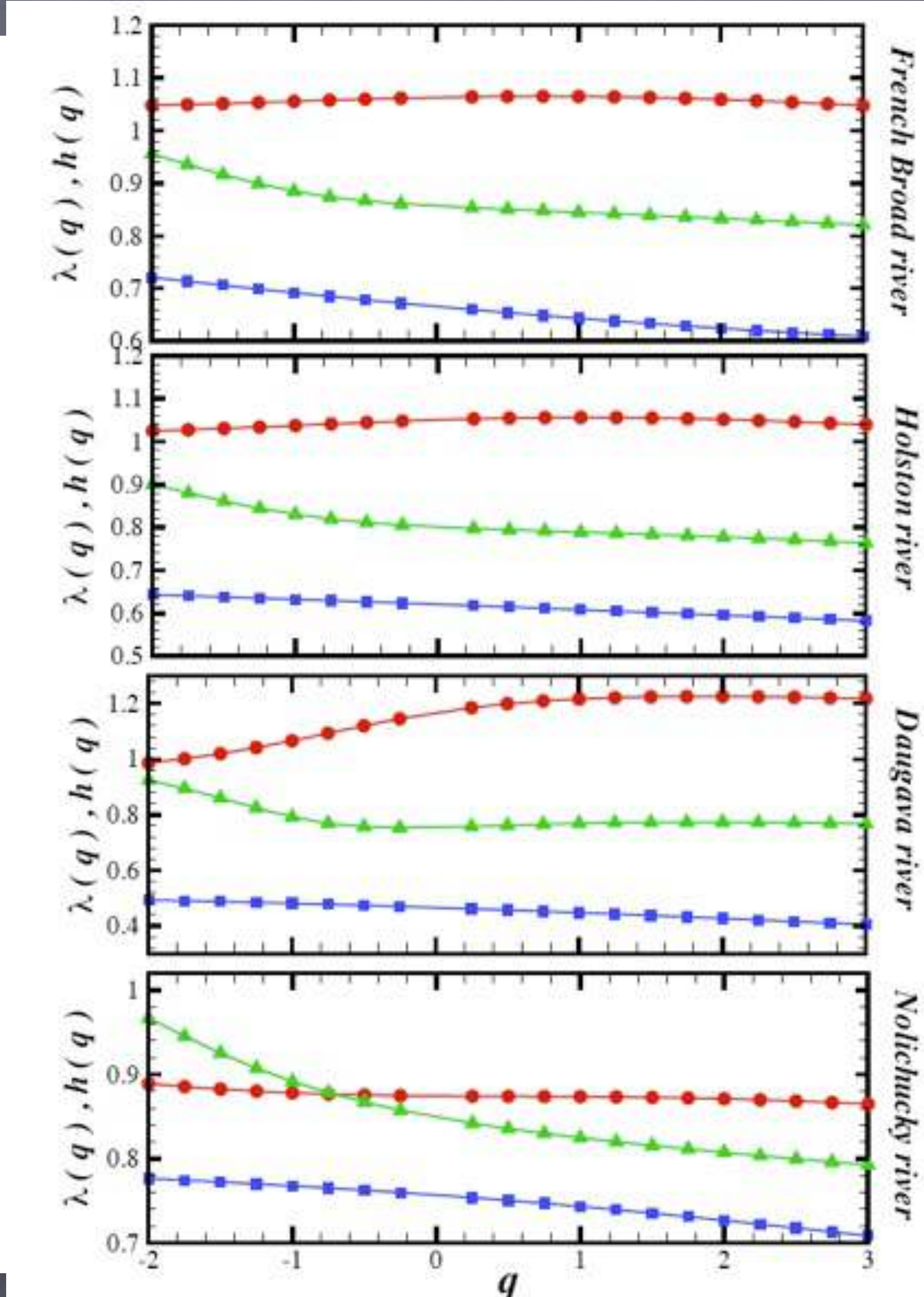

s_{1x}~ 12 months
s_{2x}~ 133 months
F_{DFA} (s)
s (month)
Sunspot
Daugava river
French Broad river
Holston river
Nolichucky river



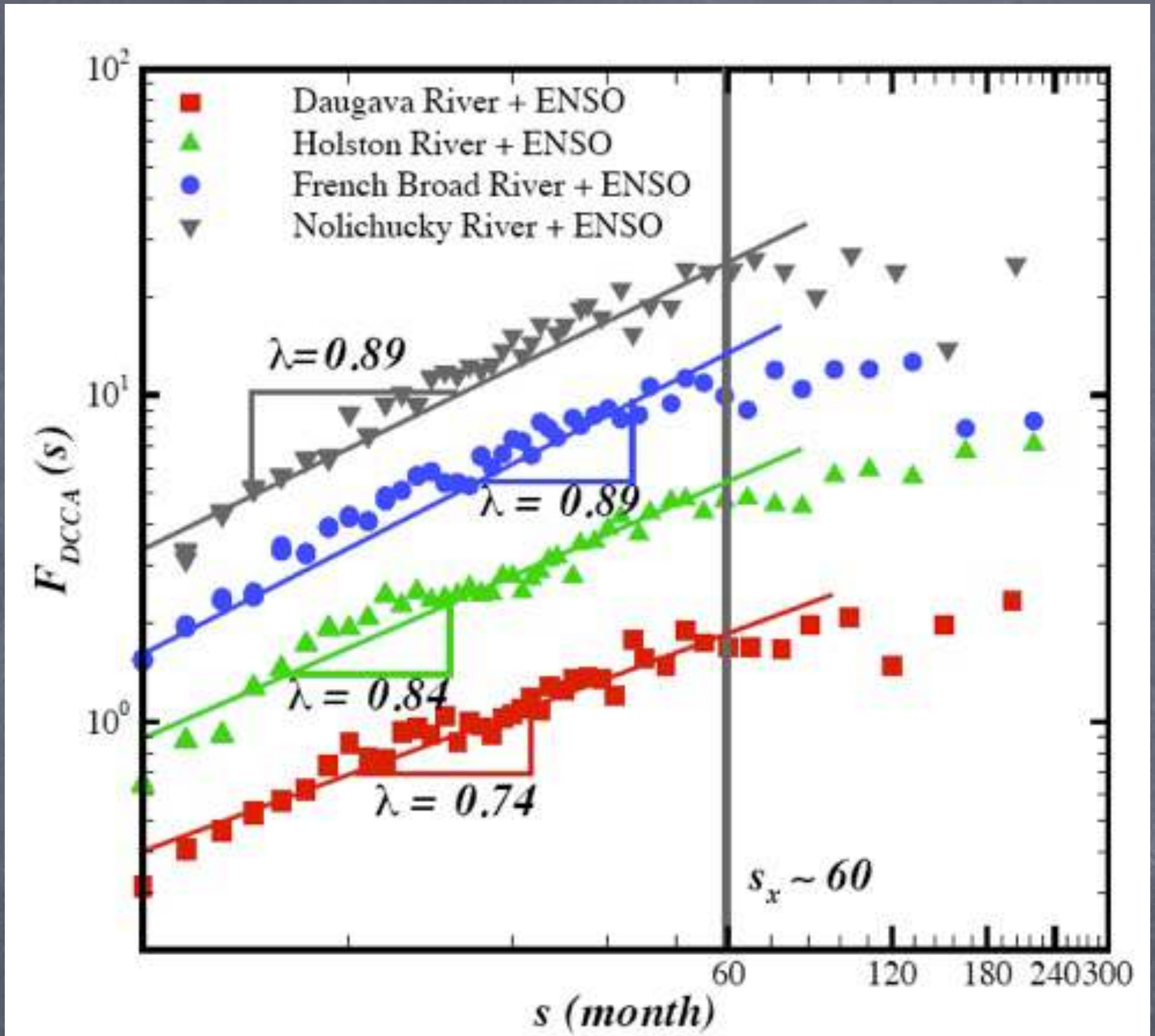


for $12\text{–}24 < s < 130$


Nolichucky River + Sunspot
French Broad River + Sunspot
Holston River + Sunspot
λ = 0.85
λ = 0.84
λ = 0.78
$F_{DCCA}(s)$
s (month)
20
15
10
5
150
300
450
600


French Broad river
Holston river
Daugava river
Nolichucky river
λ(q), h(q)
q

El Nino index3 (ENSO) and sun activity



River	λ_{ENSO}	$\gamma_\times^{ENSO}$	$\lambda_{sunspot}$	$\gamma_\times^{sunspot}$
Nolichucky	0.89 ± 0.03	0.22 ± 0.06	0.94 ± 0.01	0.12 ± 0.02
French Broad	0.89 ± 0.03	0.22 ± 0.06	0.98 ± 0.02	0.04 ± 0.02
Holston	0.85 ± 0.03	0.30 ± 0.06	0.90 ± 0.01	0.20 ± 0.02
Daugava	0.74 ± 0.03	0.52 ± 0.06	0.77 ± 0.01	0.46 ± 0.02

El nino warming the surface
La nino cooling the surface

Main results

- There is a universal behavior for river flow fluctuation during "12-24 < s < 130" months. $\lambda=1.17\pm0.04$
- There is a crossover at $s_{\times}\sim130$ months for all mentioned rivers.
- The contribution of sun activity represented by sunspot is larger than ENSO effect at least since 1950.
- Due to various values of λ's for different rivers, we conclude that beside sun activity, the geographical position, human activity, drainage network have also reasonable impact on the runoff water fluctuations

User manual for MF-DFA code written by Sadegh Movahed

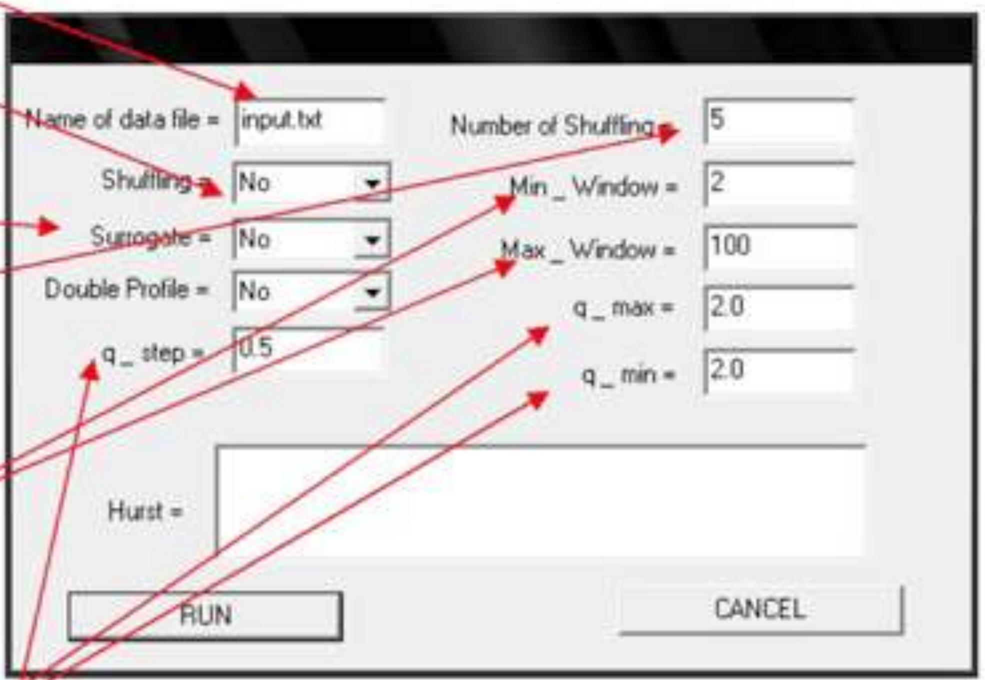
1: You should write the name of your data file in it

2: To shuffled data set you should select YES here.

3: If you want to surrogate your data, select YES for this option

4: This value shows the number of shuffling data set.

5: Here you should determine the maximum and minimum no. of windows, i.e. if you select "10" for maximum and "2" for minimum, your data set is divided to 2 up to 10 non-overlapping windows.



6: If you want to calculate just H=h(q=2) you should determine q=2., namely, q_max=q_min=2. To find the generalized Hurst exponent i.e. h(q) versus q(moment exponent), must q_min and q_max to be different. Just in this case you can find the singularity spectrum for data set.

7: Here the step of moment exponent is determined.

8: In some case, we have to use double profile for data. It is done by the proper option in my program.

The name of output files are as follows:

1) hurst.txt gives generalized Hurst exponent versus q

2) log_f_s.txt gives the ln (F(s)) versus ln(s)

3) f_s.txt gives the fluctuation function versus "s"

4) tau.txt gives classical multifractal scaling exponent

5) D.txt gives generalized multifractal dimension

6) singularity.txt gives singularity spectrum

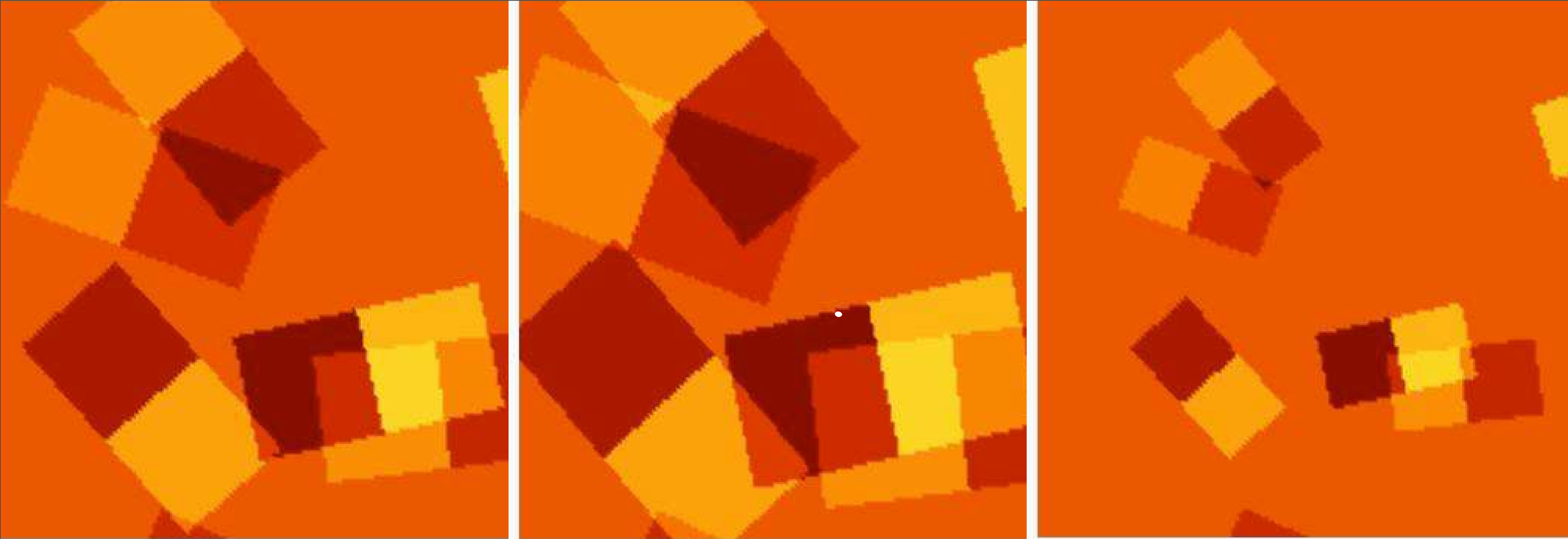
7) PDF.txt gives probability density function

Summary

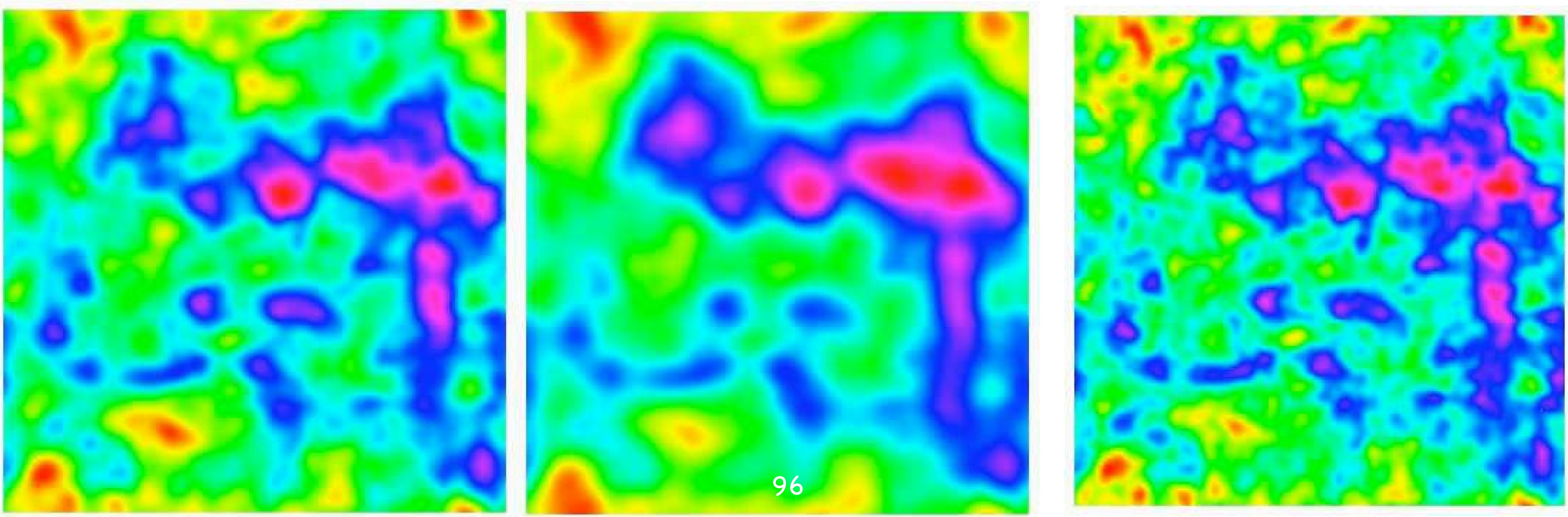
- I demonstrated the general view about fractal and multifractal time series.
- Some novel and robust methods in data analysis, in various dimensions were explored
- The relation of derived exponents and more relevant ones from complex systems point of view were developed
- I investigated the effect of various trends in methods
- An application in climate and hydrology was considered

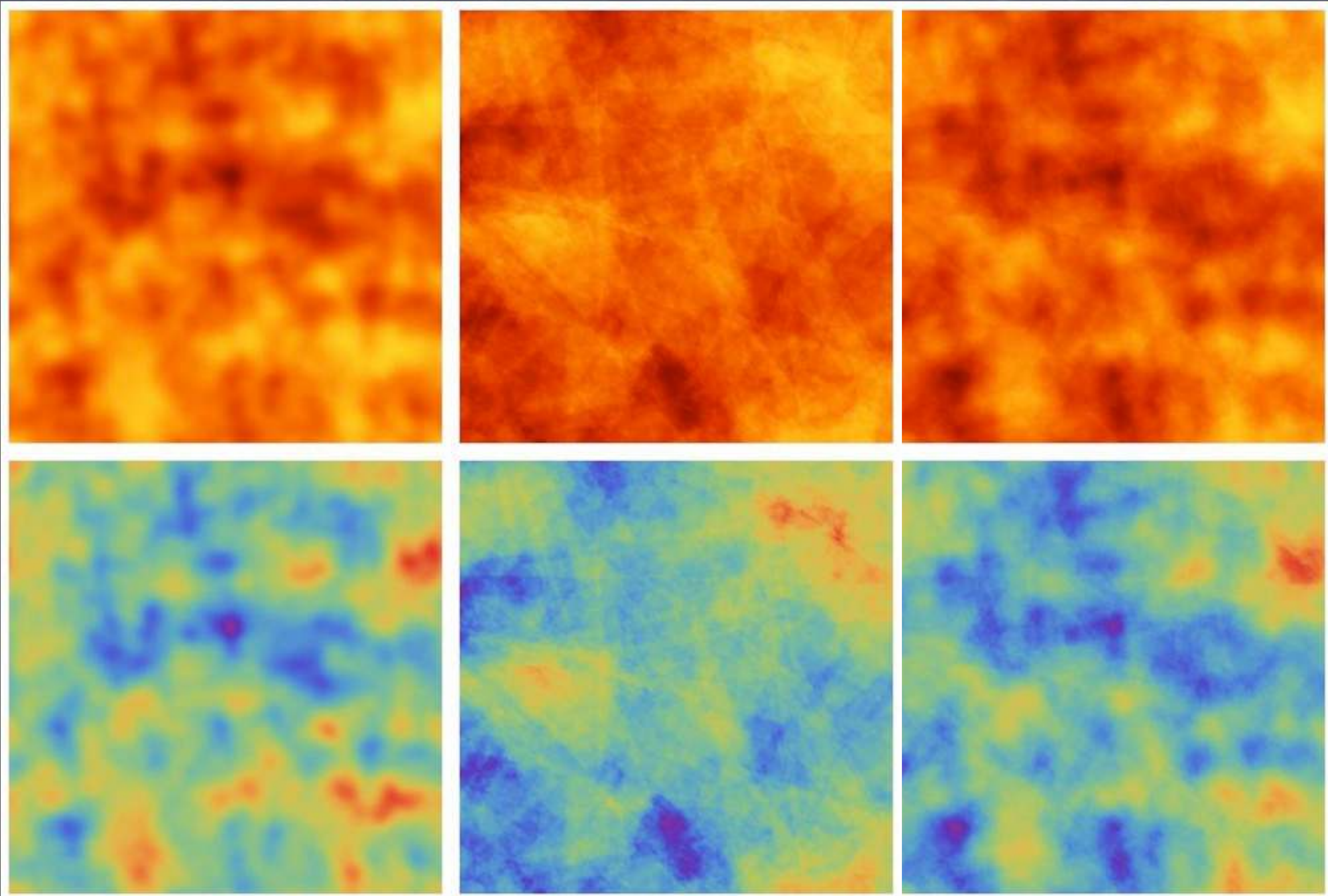
Future perspective

- Method construction (More robust and reliable), effect of trends and detrending procedures
- Applications: Due to robustness technologies for recoding various type of data set, ranging form nano-scales to MPc scales, we expect that there are many fantastic opportunities, especially in multidisciplinary sciences to get deep insight and new interpretations regarding to phenomena.



$\Omega_m = 0.27$	$\Omega_m = 0.8$	$\Omega_m = 0.3$
$\Omega_\lambda = 0.73$	$\Omega_\lambda = 0.3$	$\Omega_\lambda = 0.2$





G $\Theta = 2.5^\circ$ $R = 1'$ S GS

Thank you