بسم الله الرحمن الرحیم

[illegible] ۱۴۰۱/۸/۱۵

Ising Model

☆ مدل ساده

☆ نسبتاً قابل تعمیم است

☆ تا حدی به صورت تحلیلی قابل بررسی است ⟵ $Z = \checkmark$ تحلیلی

Part (A) 1-Dimensional Ising Model

Part (B) 2-Dimensional Ising Model

1-D Ising Model

☆ Changing Variable Method

☆ Recursive Method

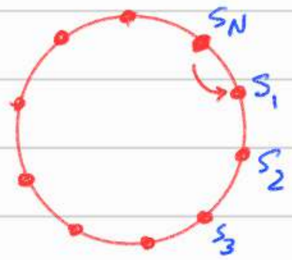
☆ Transfer Matrix Method

1. Open Boundary Condition (O) — شرط مرزی باز

2. Periodic Boundary Condition (P) — شرط مرزی تناوبی

In thermodynamical limit $O \triangleq P$

$$\overbrace{\underset{s_1}{\bullet} \quad \underset{s_2}{\bullet} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \underset{s_N}{\bullet}}^{N-\text{spins}}$$



$$\boxed{S_{N+1} = S_1}$$

Ex 1: Open Boundary Condition for 1-D Ising Model (Changing Variable)

$$\mathcal{H} = -J \sum_{i=1}^{N-1} S_i S_{i+1} - \cancel{H \sum_{i=1}^{N} S_i}$$

Cts $\qquad$ $H=0$ فرض

$$Z = \sum_{\{S=\pm1\}} e^{-\beta\mathcal{H}} = \sum_{S_1=-1}^{+1} \sum_{S_2=-1}^{+1} \cdots \sum_{S_N=-1}^{+1} e^{+\beta J \overbrace{(\underbrace{S_1S_2}_{\varrho_1} + \underbrace{S_2S_3}_{\varrho_2} + \cdots + \underbrace{S_{N-1}S_N}_{\varrho_{N-1}})}^{(N-1)\text{-Terms}}}$$

$$\varrho_1 \equiv S_1S_2 \qquad \varrho_2 \equiv S_2S_3 \qquad \cdots \qquad \varrho_{N-1} = S_{N-1}S_N \qquad S=\pm1 \longrightarrow \varrho = \pm1$$

$\uparrow\ \uparrow\ \downarrow$ ($S_1\ S_2\ S_3$) $\quad \varrho_1 = +1 \quad \varrho_2 = -1$

$\downarrow\ \downarrow\ \uparrow$ ($S_1\ S_2\ S_3$) $\quad \varrho_1 = +1 \quad \varrho_2 = -1$

$$Z_0 = 2 \sum_{\{\varrho=\pm1\}} e^{+\beta J \overbrace{(\varrho_1 + \varrho_2 + \varrho_3 + \cdots + \varrho_{N-1})}^{(N-1)\text{-terms}}}$$

$$\boxed{Z_0 = 2\left(2\cosh \beta J\right)^{N-1}}$$

تابع پارش مدل آیزینگ $H=0$ یک‌بعدی
با شرط مرزی باز

Ex 2: Periodic Boundary Condition 1-D Ising Model.

$$\boxed{S_{N+1} = S_1} \qquad \mathcal{H} = -J \sum_{i=1}^{N} S_i S_{i+1}$$

$$S_i S_{i+1} = \varrho_i$$

$$Z_p = 2\sum_{\{\sigma=\pm1\}} e^{\beta J(\sigma_1+\sigma_2+\cdots+\sigma_{N-1}) + \beta J S_N \overset{S_1}{\widetilde{S}_{N+1}}}$$

$$Z_p = 2\sum_{\{\sigma=\pm1\}} e^{\beta J(\overbrace{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{N-1}}^{(N-1)\text{-terms}}) + \beta J \underbrace{S_1S_2}_{\sigma_1}\overbrace{S_2S_3}^{\sigma_2}\underbrace{S_3\,\cdot}_{\sigma_3}\cdots\underbrace{S_{N-1}S_N}_{\sigma_{N-1}}}$$

$$= 2\sum_{\{\sigma=\pm1\}} e^{\beta J(\overbrace{\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_{N-1}}^{(N-1)\text{-terms}})} \underbrace{e^{\beta J\overbrace{\sigma_1\sigma_2\cdots\sigma_{N-1}}^{(N-1)\text{-terms}}}}_{\sum_{n=0}^{\infty}\frac{(\beta J)^n}{n!}(\sigma_1\cdots\sigma_{N-1})^n}$$

$$= 2\sum_{n=0}^{\infty}\frac{(\beta J)^n}{n!}\left[\sum_{\sigma=\pm1} e^{\beta J\sigma}\sigma^n\right]^{N-1}$$

$$= 2\sum_{n=0}^{\infty}\frac{(\beta J)^n}{n!}\left\{e^{\beta J}+(-1)^n e^{-\beta J}\right\}^{N-1}$$

$$Z_p = \underbrace{2\underbrace{\sum_{n=0}^{\infty}\frac{(\beta J)^{2n}}{(2n)!}}_{e^{\beta J}}\{\underbrace{e^{\beta J}+e^{-\beta J}}_{2\cosh\beta J}\}^{N-1}}_{\text{even-}n} + \underbrace{2\sum_{n=0}^{\infty}\frac{(\beta J)^{2n+1}}{(2n+1)!}\{\underbrace{e^{\beta J}-e^{-\beta J}}_{2\sinh\beta J}\}^{N-1}}_{\text{odd-}n}$$

$$\boxed{Z_p = (2\cosh\beta J)^N + (2\sinh\beta J)^N}$$

$$\boxed{Z_0 = 2(2\cosh\beta J)^{N-1}}$$

$$\boxed{\lim_{N\to\infty} Z_p = Z_0}$$

Ex3: 1-D Ising Model (Recursive method)

$$Z_0 = \sum_{\{S=\pm1\}} e^{\beta J(S_1S_2+S_2S_3+\cdots+S_{N-1}S_N)}$$

$$= \sum_{S_1=-1}^{+1}\sum_{S_2=-1}^{+1}\cdots\underbrace{\sum_{S_N=-1}^{+1} e^{\beta J S_{N-1}\overset{+1\text{ or }-1}{S_N}}}_{2\cosh\beta J S_{N-1}}\; e^{\beta J(S_1S_2+S_2S_3+\cdots+S_{N-2}S_{N-1})}$$

$$Z_0 = \overset{\sum_{S_{N-1}}}{Z_{N-1}}\,(2\cosh\beta J\,\cancel{S_{N-1}})$$

$$Z_N = Z_{N-1}\,(2\cosh\beta J) \longrightarrow \boxed{Z_N = 2\,(2\cosh\beta J)^{N-1}}$$

Ex4: Transfer Matrix method for periodic condition

$$Z_p = \sum_{\{S=\pm1\}} e^{\beta J(S_1S_2+S_2S_3+\cdots+S_{N-1}S_N+S_NS_1)}$$

$$e^{\beta J S_i S_{i+1}} \equiv \langle S_i|\hat{T}|S_{i+1}\rangle \equiv T_{i,i+1}$$

$$|S\rangle = \begin{cases} \begin{pmatrix}1\\0\end{pmatrix} = |+\rangle \\ \begin{pmatrix}0\\1\end{pmatrix} = |-\rangle \end{cases}$$

$$\hat{T} = \begin{pmatrix} \langle +|\hat{T}|+\rangle & \langle +|\hat{T}|-\rangle \\ \langle -1|\hat{T}|+\rangle & \langle -1|\hat{T}|-1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta J} \end{pmatrix}$$

$$Z_p = \sum_{\{s\}} \overbrace{\langle s_1|\hat{T}|s_2\rangle}^{e^{\beta J s_1 s_2}} \underbrace{\rangle\langle}_{\hat{1}} \overbrace{s_2|\hat{T}|s_3\rangle}^{\hat{1}} \cdots \overbrace{\langle s_N|\hat{T}|s_1\rangle}$$

$$= \sum_{s_1=\pm1} \langle s_1|\hat{T}^N|s_1\rangle = \mathrm{Tr}[\hat{T}^N]$$

$$Z_p = \mathrm{Tr}[\hat{T}^N] = \mathrm{Tr}(U^+ U \hat{T}^N) = \mathrm{Tr}(U^T \hat{T}^N U)$$

$$= \mathrm{Tr}(\tilde{T}^N)$$

$$= \mathrm{Tr}\begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix}$$

$$Z_p = \lambda_+^N + \lambda_-^N$$

$$U = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$H \neq 0 \longrightarrow \mathcal{H} = -J\sum S_i S_{i+1} - H\sum S_i$$

$$= -J\sum S_i S_{i+1} - \frac{H}{2}\sum_i (S_i + S_{i+1})$$

$$\hat{T} = \langle S_i|T|S_{i+1}\rangle = \begin{bmatrix} e^{\beta J + \beta H} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta J - \beta H} \end{bmatrix}$$

$$\tilde{T} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} \longrightarrow \tilde{T}^N = \begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix}$$

$$\lambda_\pm = e^{\beta J}\left[\cosh\beta H \pm \sqrt{\sinh^2_\beta H + e^{-4\beta J}}\right]$$

$$\lambda_+ > \lambda_-$$

$$Z_p = T_r(\hat{T}^N) = T_r(\tilde{T}^N) = \lambda_+^N + \lambda_-^N$$

$$= \lambda_+^N\left[1 + \frac{\lambda_-^N}{\lambda_+^N}\right] = \lambda_+^N\left[1 + (\underbrace{\frac{\lambda_-}{\lambda_+}}_{<1})^N\right]$$

$$\boxed{\lim_{N\to\infty} Z_p \simeq \lambda_+^N}$$

also $f = -\frac{1}{N} K_B T \ln Z$

$$M = -\frac{\partial f}{\partial H} = \frac{\sinh(\beta H)}{\sqrt{\sinh^2(\beta H) + e^{-4\beta J}}}$$

$\lim_{\substack{H\to 0\\ \beta J = \text{finite}}} M = 0$ and $\lim_{\substack{H\to 0\\ \beta J\to\infty}} M = \text{cts}$

Ex5 : Correlation function $G(i,j) = \langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle = ?$

$H=0$

$$\mathcal{H} = -J\sum_{ij} S_i S_j = -\sum J_{ij} S_i S_j = -\sum J_i S_i S_{i+1} \qquad J_{ij} = +J$$

$$\langle S_i S_{i+1}\rangle = \sum_{\{S\}} S_i S_{i+1} \frac{e^{-\beta\mathcal{H}}}{Z}$$ Probability

$$\langle S_i S_{(i+1)}\rangle = + \frac{\partial \ln Z}{\partial \beta J_i}$$

$$\langle S_i S_{i+2}\rangle = \frac{1}{Z}\frac{\partial}{\partial\beta J_i}\frac{\partial}{\partial\beta J_{i+1}} Z = \langle S_i \overbrace{S_{i+1} S_{i+1}}^{+1} S_{i+2}\rangle = \sum S_i S_{i+1} S_{i+1} S_{i+2} \frac{e^{-\beta\mathcal{H}}}{Z}$$

ثابت کردیم

$$Z = 2\prod_{i=1}^{N-1}(2\cosh\beta J_i)$$

$T > 0$, 1-D $\rightarrow \langle S \rangle = 0$

$$\langle S_i \, S_{i+r} \rangle = (\tanh \beta J)^r$$

$$G(i, i+r) = (\tanh \beta J)^r \equiv \underbrace{e^{-r/\xi}}_{\text{درخواست}} \rightarrow \boxed{\xi = \frac{1}{\ln \coth \beta J}}$$

$$S_i \xrightarrow{\;r\;} S_{i+r}$$

Ex 6. Correlation function : Transfer Matrix approach

$$G(i,j) = \langle S_i \, S_j \rangle - \langle S_i \rangle \langle S_j \rangle = ?$$

$$\mathcal{H} = -J \sum_{i=1}^{N} S_i \, S_{i+1} \quad , H = 0$$

$$\langle S_i \rangle = \frac{1}{Z} \sum_{S_1 = \pm 1} \sum_{S_2 = \pm 1} \cdots \sum_{S_N = \pm 1} S_i \prod_{i=1}^{N} e^{\beta J S_i S_{i+1}}$$

$$= \frac{1}{Z} \sum_{\{S\}} S_i \langle S_1 | \hat{T} | S_2 \rangle \langle S_2 | T | S_3 \rangle \cdots$$

$$= \frac{1}{Z} \sum_{\{S\}} \underbrace{\hat{T}_{S_1 S_2} \hat{T}_{S_2 S_3} \; T_{S_{i-1} S_i}}_{(i-1)\text{-Terms}} \; S_i \; \underbrace{T_{S_i S_{i+1}} \cdots T_{S_N S_1}}_{(N-(i-1))\text{-Terms}}$$

$$= \frac{1}{Z} \operatorname{Tr} \left[\hat{T}^{(i-1)} \hat{S} \, T^{N-(i-1)} \right]$$

$$= \frac{1}{Z} \operatorname{Tr} \left[\hat{S} \hat{T}^N \right] \qquad \hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{Z} \operatorname{Tr} \left[\underbrace{U^T \hat{T}^N U}_{\tilde{T}^N} \underbrace{U^T \hat{S} U}_{S' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \right]$$

$$= \frac{1}{Z} \operatorname{Tr}\left[\begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$H=0$
$T>0$

$$\langle S_i \rangle = \frac{1}{Z} \operatorname{Tr}\begin{bmatrix} 0 & \lambda_+^N \\ \lambda_-^N & 0 \end{bmatrix} = 0$$

دلیل به دست آمدن این نتیجه این است که ما هیچ فرضی مبنی بر اینکه $T=0$ باشد نکردیم ولی حتی عناصر ماتریس T باید محدود باشند در حالی که در حالت $T \to 0$ یعنی $\beta \to \infty$ دیگر عبارت بالا درست نیست لذا نتایج بالا صرفاً برای حالت β محدود یعنی $T \neq 0$ کار می‌کند که به واقع انتظار هم داریم که درست

در حالت $H=0$ و

$$\langle S_i S_j \rangle = \langle S_i S_{i+r} \rangle = \frac{1}{Z} \operatorname{Tr}\left[\hat{T}^{i-1}\hat{S}\,\underbrace{\hat{T}^r}_{\hat{T}^r}\, S\, \underbrace{\hat{T}^{N-(i-r)+1}}_{\hat{T}^{N-r}}\right]$$

و حالت یک‌بعدی $M=0$ باشد ($T>0$)
برای اینکه نتیجه کامل‌تری داشته باشیم
بهتر است از ابتدا $H \neq 0$ را در نظر بگیریم و بعد
$H \to 0$ میل دهیم

$$= \frac{1}{Z} \operatorname{Tr}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_+^r & 0 \\ 0 & \lambda_-^r \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_+^{N-r} & 0 \\ 0 & \lambda_-^{N-r} \end{pmatrix}\right]$$

$$\langle S_i S_{i+r} \rangle = \frac{\lambda_+^{N-r}\lambda_-^r + \lambda_-^{N-r}\lambda_+^r}{\lambda_+^N + \lambda_-^N}$$

$$\lim_{N\to\infty} \langle S_i S_{i+1} \rangle \simeq \frac{\lambda_+^{N-r}\lambda_-^r + \lambda_-^{N-r}\lambda_+^r}{\lambda_+^N}$$

$$\simeq \left(\frac{\lambda_-}{\lambda_+}\right)^r + \underbrace{\left(\frac{\lambda_-}{\lambda_+}\right)^{N-r}}_{\to 0}$$

$N \gg r$

$$\lim_{\substack{N\to\infty \\ N \gg r}} \langle S_i S_{i+r} \rangle \simeq \left(\frac{\lambda_-}{\lambda_+}\right)^r \equiv e^{-r/\xi}$$

$$\xi \equiv \frac{1}{\ln\left[\frac{\lambda_+}{\lambda_-}\right]}$$

For $T=0$, $H=0$ $\quad \lambda_+ = \lambda_- \longrightarrow \xi = \infty$

نظم بلندبرد داریم ← فاز منظم داریم

For $T>0$ $\quad \lambda_+ > \lambda_- \longrightarrow \xi =$ محدود است

نظم کوتاه برد داریم ← فاز منظم نداریم

جلسه ۱۷/۸/۱۴۰۱

بسم الله الرحمن الرحیم

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i$$

2-Dimensional Ising Model.

☆ Perturbative Method → low temperature $K_B T < J$; → High temperature $K_B T > J$

1946 Lars Onsager $\quad H=0$

$$* \quad T_c = \frac{2J}{K_B \ln(Z-1)}$$

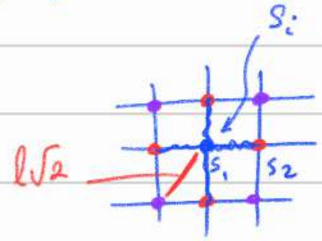
Z → coordinate No

EX1: High temperature $\quad K_B T > J \quad \boxed{\text{2-D}}$

$H=0$

$$\mathcal{H} = -J \sum S_i S_j \longrightarrow Z = \sum_{\{S\}} \underbrace{e^{+\beta J \sum_{\langle ij \rangle} S_i S_j}}$$

$$Z = \sum_{\{S\}} \prod_{\substack{\text{link} \\ \langle ij \rangle}} e^{+\beta J S_i S_j}$$



$S_i = \pm 1$