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Answer to Exercise set 7

1. Non linear Langvin equation is:

$$\dot{v}(t) = h(v,t) + g(v,t)\eta(t)$$

Keramers-Moyal coefficients comes form:

$$D^{(n)}(v,t) = \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\langle\ [\xi(t+\tau)-v]^n\ \rangle\Big|_{\xi(t)=v}$$

We should calculate below for two method:

$$\xi(t+\tau) - v = \int_t^{t+\tau}\Big[h(\xi(t'),t') + g(\xi(t'),t')\eta(t')\Big]dt' \tag{1}$$

We know Weiner function define as:

$$w(\tau) = W(t+\tau) - W(t) = \int_t^{t+\tau}\Gamma(t')dt'$$

With these properties:

$$w(0) = 0$$
$$\langle\ w(\tau)\ \rangle = 0$$
$$\langle\ w(\tau_1)w(\tau_2)\ \rangle = 2min(\tau_1,\tau_2)$$

We can calculate 1 with Itô and Stratonovich with these definition:

$$A_{\substack{I\\S}} \overset{\text{(I)}}{\underset{\text{(S)}}{=}} \int_0^{\tau}\Phi(w(\tau'),\tau')dw(\tau')$$

$$A_I = \lim_{\Delta\to 0}\sum_{i=0}^{N-1}\Phi(w(\tau_i),\tau_i)[w(\tau_{i+1}) - w(\tau_i)]$$
$$A_s = \lim_{\Delta\to 0}\sum_{i=0}^{N-1}\Phi\Big(\frac{w(\tau_{i+1})+w(\tau_i)}{2},\frac{\tau_i+\tau_{i+1}}{2}\Big)[w(\tau_{i+1}) - w(\tau_i)],$$

With Δ:

$$\Delta = max(\tau_{i+1}-\tau_i) \qquad 0 = \tau_0 < \tau_1 ... < \tau_N = \tau$$

Now we use Taylor expansion in 1 we have:

$$\begin{aligned}\xi(t+\tau) - v &= \int_t^{t+\tau}\Big[h(\xi(t'),t') + g(\xi(t'),t')\eta(t')\Big]dt' \\ &= \int_t^{t+\tau}h(\xi(t'),t')dt' + \int_t^{t+\tau}g(\xi(t'),t')\eta(t')dt'\end{aligned} \tag{2}$$

We use:

$$\begin{aligned} h(\xi(t'),t') &= h(v,t') + h'(v,t')(\xi(t')-v) + \dots \\ g(\xi(t'),t') &= g(v,t') + g'(v,t')(\xi(t')-v) + \dots \end{aligned} \tag{3}$$

Put 3 in 2 we can get:

$$\begin{aligned} \xi(t+\tau) - v &= \int_t^{t+\tau} \Big[h(v,t') + h'(v,t')(\xi(t')-v) + \dots \\ &\qquad g(v,t')\eta(t') + g'(v,t')(\xi(t')-v)\eta(t') + \dots \Big] dt' \\ &= \int_t^{t+\tau} h(v,t')dt' + \int_t^{t+\tau} h'(v,t')(\xi(t')-v)dt' + \dots \\ &+ \int_t^{t+\tau} g(v,t')\eta(t')dt' + \int_t^{t+\tau} g'(v,t')(\xi(t')-v)\eta(t')dt' + \dots \end{aligned} \tag{4}$$

In here we use again of 1 and expand $\xi(t') - v$ we have:

$$\begin{aligned} \xi(t') - v &= \int_t^{t'} \Big[h(v,t'') + h'(v,t'')(\xi(t'')-v) + \dots \Big] dt'' \\ &+ \int_t^{t'} \Big[g(v,t'') + g'(v,t'')(\xi(t'')-v) + \dots \Big] \eta(t'')dt'' \end{aligned}$$

Put it in 4 we can get:

$$\begin{aligned} \xi(t+\tau) - v &= \int_t^{t+\tau} h(v,t')dt' + \int_t^{t+\tau} h'(v,t') \int_t^{t'} h(v,t'')dt''dt' \\ &+ \int_t^{t+\tau} h'(v,t') \int_t^{t'} g(v,t'')\eta(t'')dt''dt' + \dots \\ &+ \int_t^{t+\tau} g(v,t')\eta(t')dt' + \int_t^{t+\tau} g'(v,t') \int_t^{t'} h(v,t'')\eta(t')dt''dt' \\ &+ \int_t^{t+\tau} g'(v,t') \int_t^{t'} g(v,t'')\eta(t'')\eta(t')dt''dt' + \dots \end{aligned} \tag{5}$$

Take time average:

$$\begin{aligned} \langle\, \xi(t+\tau) - v \,\rangle &= \int_t^{t+\tau} h(v,t')dt' + \int_t^{t+\tau} h'(v,t') \int_t^{t'} h(v,t'')dt''dt' \\ &+ \int_t^{t+\tau} g'(v,t') \int_t^{t'} g(v,t'')\eta(t'')\eta(t')dt''dt' + \dots \end{aligned} \tag{6}$$

Now for above we can get different result using Itô and Stratonovich method because:

$$\begin{aligned} \langle\, \xi(t+\tau) - v \,\rangle &= h(v,t+\tau)\tau + h'(v,t+\tau)h(v,t+\tau)\tau^2 \\ &+ g'(v,t+\tau)g(v,t+\tau)\Big\langle \int_0^{\tau} w(\tau')dw(\tau') \,\Big\rangle \end{aligned}$$

Right angel and left angel in right calculated by two method differently:

$$\left\langle (I) \int_0^\tau w(\tau')dw(\tau') \right\rangle = \left\langle \sum_{i=0}^{N-1} w(\tau_i)[w(\tau_{i+1}) - w(\tau_i)] \right\rangle$$
$$= \sum_{i=0}^{N-1} [\langle\, w(\tau_i)w(\tau_{i+1})\,\rangle - \langle\, w(\tau_i)w(\tau_i)\,\rangle]$$
$$= \sum_{i=0}^{N-1} (2\tau_i - 2\tau_i) = 0$$

$$\left\langle (S) \int_0^\tau w(\tau')dw(\tau') \right\rangle = \frac{1}{2}\left\langle \sum_{i=0}^{N-1} [w(\tau_i) + w(\tau_{i+1})][w(\tau_{i+1}) - w(\tau_i)] \right\rangle$$
$$= \frac{1}{2}\sum_{i=0}^{N-1} [\langle w(\tau_{i+1})w(\tau_{i+1})\rangle - \langle w(\tau_i)w(\tau_i)\rangle]$$
$$= \frac{1}{2}\sum_{i=0}^{N-1} (2\tau_{i+1} - 2\tau_i) = \tau$$

So we have:

$$(I): \quad D^{(1)} = h(v,t)$$
$$(S): \quad D^{(1)} = h(v,t) + g'(v,t)g(v,t)$$

For second Kramers-Moyal coeff comes form 5, we have:

$$(\xi(t+\tau) - v)^2 = \Bigg(\int_t^{t+\tau} h(v,t')dt' + \int_t^{t+\tau} h'(v,t') \int_t^{t'} h(v,t'')dt''dt'$$
$$+ \int_t^{t+\tau} h'(v,t') \int_t^{t'} g(v,t'')\eta(t'')dt''dt' + \dots$$
$$+ \int_t^{t+\tau} g(v,t')\eta(t')dt' + \int_t^{t+\tau} g'(v,t') \int_t^{t'} h(v,t'')\eta(t')dt''dt'$$
$$+ \int_t^{t+\tau} g'(v,t') \int_t^{t'} g(v,t'')\eta(t'')\eta(t')dt''dt' + \dots \Bigg)^2$$

With taking time average we can get:

$$\langle\, (\xi(t+\tau) - v)^2 \,\rangle = \Bigg\langle \Big(\int_t^{t+\tau} h(v,t')dt' + \int_t^{t+\tau} h'(v,t') \int_t^{t'} h(v,t'')dt''dt'$$
$$+ \dots + \int_t^{t+\tau} g(v,t')\eta(t')dt'$$
$$+ \int_t^{t+\tau} g'(v,t') \int_t^{t'} g(v,t'')\eta(t'')\eta(t')dt''dt' + \dots \Big)^2 \Bigg\rangle$$

We can show that only below term can survive:

$$\langle\, (\xi(t+\tau) - v)^2 \,\rangle = g^2(v+t+\tau)\langle w(\tau)w(\tau)\rangle + \Big(g'(v,t+\tau)g(v,t+\tau) \int_0^\tau w(\tau')dw(\tau') \Big)^2$$

Second term on right is zero in It Itô method and in Stratonovich method in order of τ^2 in both case second Kramers-Moyal coeff is:

$$D^{(2)} = \frac{1}{2\tau} g^2(v, t+\tau) 2\tau = g^2(v, t+\tau)$$

So:

$$\begin{aligned} (I): &\quad D^{(2)} = g^2(v,t) \\ (S): &\quad D^{(2)} = g^2(v,t) \end{aligned}$$

Finally we can write none linear Langvin equation for both methods:

$$\begin{aligned} (I): &\quad \dot{v}(t) = D^{(1)} + \sqrt{D^{(2)}}\eta(t) = h(v,t) + \sqrt{g^2(v,t)}\eta(t) \\ (S): &\quad \dot{v}(t) = D^{(1)} + \sqrt{D^{(2)}}\eta(t) = h(v,t) + g'(v,t)g(v,t) + \sqrt{g^2(v,t)}\eta(t) \end{aligned}$$

2. We should show that $\bar{D}^i$ is contravariant vector so:

$$\frac{\partial x'k}{\partial x^i}\bar{D}^i = \frac{\partial x'k}{\partial x^i}D_i - \frac{\partial x'k}{\partial x^i}\sqrt{Det}\frac{\partial}{\partial x^j}\frac{\bar{D}^{ij}}{\sqrt{Det}} \tag{7}$$

We use:

$$\frac{\partial}{\partial x^j} = \frac{1}{J}\frac{\partial}{\partial x'^r}\frac{\partial x'^r}{\partial x^j}J \tag{8}$$

$$\begin{aligned} D'_k &= \frac{\partial x'_k}{\partial x_i}D_i + \frac{\partial^2 x'_k}{\partial x_i \partial x_j}D^{ij} \qquad if \qquad \left(\frac{\partial x'_k}{\partial t}\right)_x = 0 \\ \frac{\partial x'_k}{\partial x_i}D_i &= D'_k - \frac{\partial^2 x'_k}{\partial x_i \partial x_j}D^{ij} \end{aligned} \tag{9}$$

Put 8 and 9 in 7, we have:

$$\begin{aligned} \frac{\partial x'k}{\partial x^i}\bar{D}^i &= D'_k - \frac{\partial^2 x'_k}{\partial x_i \partial x_j}\bar{D}^{ij} - \frac{\partial x'k}{\partial x^i}\frac{\sqrt{Det}}{J}\frac{\partial}{\partial x'^r}\left(\frac{\partial x'^r}{\partial x^j}J\frac{\bar{D}^{ij}}{\sqrt{Det}}\right) \\ &= D'_k - \frac{\partial^2 x'_k}{\partial x_i \partial x_j}\bar{D}^{ij} - \sqrt{Det'}\frac{\partial}{\partial x'^r}\frac{\partial x'k}{\partial x^i}\frac{\partial x'^r}{\partial x^j}J\frac{\bar{D}^{ij}}{\sqrt{Det}} \\ &\quad + \sqrt{Det'}\left(\frac{\partial}{\partial x'^r}\frac{\partial x'k}{\partial x^i}\right)\frac{\partial x'^r}{\partial x^j}\frac{\bar{D}^{ij}}{\sqrt{Det'}} \end{aligned} \tag{10}$$

In here we use:

$$\begin{aligned} \sqrt{Det'}\frac{\partial}{\partial x'^r}\left(\frac{\partial x'k}{\partial x^i}\frac{\partial x'^r}{\partial x^j}J\frac{\bar{D}^{ij}}{\sqrt{Det}}\right) &= \frac{\partial x'k}{\partial x^i}\frac{\sqrt{Det}}{J}\frac{\partial}{\partial x'^r}\left(\frac{\partial x'^r}{\partial x^j}J\frac{\bar{D}^{ij}}{\sqrt{Det}}\right) \\ &\quad + \sqrt{Det'}\frac{\partial}{\partial x'^r}\left(\frac{\partial x'k}{\partial x^i}\right)\frac{\partial x'^r}{\partial x^j}\frac{\bar{D}^{ij}}{\sqrt{Det'}} \end{aligned}$$

Last term of 10 is:

$$\begin{aligned} \sqrt{Det'}\frac{\partial}{\partial x'^r}\left(\frac{\partial x'k}{\partial x^i}\right)\frac{\partial x'^r}{\partial x^j}\frac{\bar{D}^{ij}}{\sqrt{Det'}} &= \frac{\partial}{\partial x'^r}\left(\frac{\partial x'k}{\partial x^i}\right)\frac{\partial x'^r}{\partial x^j}\bar{D}^{ij} \\ &= \frac{\partial^2 x'k}{\partial x'^r \partial x^i}\frac{\partial x'^r}{\partial x^j}\bar{D}^{ij} \\ &= \frac{\partial^2 x'k}{\partial x^i \partial x^j}\bar{D}^{ij} \end{aligned}$$

Above and second term cancel each other so:

$$\begin{aligned} \frac{\partial x'k}{\partial x^i}\bar{D}^i &= D'_k - \sqrt{Det'}\frac{\partial}{\partial x'^r}\frac{\partial x'k}{\partial x^i}\frac{\partial x'^r}{\partial x^j}\frac{\bar{D}^{ij}}{\sqrt{Det'}} \\ &= D'_k - \sqrt{Det'}\frac{\partial}{\partial x'^r}J\frac{\bar{D}^{kl}}{\sqrt{Det}} \end{aligned}$$

So this is invariant under contravariant transformation.

3. We use

$$\frac{\partial}{\partial x^i} = \frac{1}{J}\,\frac{\partial}{\partial x'^k}\,\frac{\partial x'^k}{\partial x^i}\,J$$

Put it in relation:

$$\begin{aligned}
\bar{S}^i_{;i} &= \sqrt{Det}\frac{\partial}{\partial x^i}\frac{\bar{S}^i}{\sqrt{Det}} \\
&= \frac{\sqrt{Det}}{J}\,\frac{\partial}{\partial x'^k}\,\frac{\partial x'^k}{\partial x^i}\,\frac{J}{\sqrt{Det}}\bar{S}^i \\
&= \sqrt{Det'}\,\frac{\partial}{\partial x'^k}\,\frac{\partial x'^k}{\partial x^i}\,\frac{\bar{S}^i}{\sqrt{Det'}}
\end{aligned}$$

We know that $\left(\partial x'^k/\partial x^i\right)\bar{S}^i = \bar{S}'^k$, so:

$$\begin{aligned}
\bar{S}^i_{;i} &= \sqrt{Det'}\,\frac{\partial}{\partial x'^k}\,\frac{\bar{S}^i}{\sqrt{Det'}} \\
&= \bar{S}'^k_{;k}
\end{aligned}$$

This quantity behave like scaler.