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Answer to Exercise set 5

1. This integral can NOT be calculated analytically unless in some special α, but one can approximate $p(s)$ in large s like [1]:

$$p(s) \propto \frac{1}{|s|^{1+\beta}}$$

This is heavy tailed distribution function with $0 < \beta < 2$ and $\alpha = 1 + \beta$. We can calculate characteristic function analytically with a trick, so we have:

$$\mathcal{Z}_s(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} p(x)dx$$

$$\begin{aligned}\mathcal{Z}_s(\lambda) &= 1 - \big(1 - \mathcal{Z}_s(\lambda)\big)\\ {}^2 &= 1 - \int_{-\infty}^{\infty} \big(1 - \cos\lambda x\big)p(x)dx\end{aligned}$$

Use change of variable $y = \lambda x$ then:

$$\begin{aligned}\int_{-\infty}^{\infty} \big(1-\cos\lambda x\big)p(x)dx &= \frac{1}{k}\int_{-\infty}^{\infty}\big(1-\cos y\big)p(\frac{y}{\lambda})dy\\ &\cong \frac{1}{k}\int_{-\infty}^{\infty}\big(1-\cos y\big)\frac{1}{|y/\lambda|^{1+\beta}}dy\\ &= |\lambda|^{\beta}\int_{-\infty}^{\infty}\frac{\big(1-\cos y\big)}{|y|^{1+\beta}}dy\\ {}^3 &= A|\lambda|^{\beta}\end{aligned}$$

So characteristic function is:

$$\mathcal{Z}_s(\lambda) = 1 - A|\lambda|^{\beta} + ... = 1 - A|\lambda|^{\alpha-1} + ... \tag{1}$$

We can get inverse Fourier transformation to find p(x) like this:

$$\begin{aligned}p(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{-i\lambda x}\Big[\mathcal{Z}_s(\lambda)\Big]^N\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{-i\lambda x}\Big[1 - A|\lambda|^{\alpha-1} + ...\Big]^N\\ {}^4 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{-i\lambda x} e^{-NA|\lambda|^{\alpha-1}}\end{aligned}$$

[1] You should take absolute value otherwise PDF became undefined fo s<0.
[2] Sin integral is zero because of boundary symmetry and odd function.
[3] for $\beta < 2$ integral is a constant:

$$A = \int_{-\infty}^{\infty}\frac{(1-\cos y)}{|y|^{1+\beta}}dy = \frac{\pi}{\Gamma(1+\beta)\sin(\pi\beta/2)}$$

Characteristic function became $\mathcal{Z}_{\alpha-1}(\lambda) = e^{-NA|\lambda|^{\alpha-1}}$ and this is called Levy distribution for $0 < \alpha - 1 < 2$. [5] For $\alpha = 2$ you can calculate integral, this PDF is called Caushy distribution [6] . This distribution doesn't have well-defined moments and variance.

[4] For $N \gg 1$ and $\lambda \ll 1$ you can approximate

$$\left[1 - A|\lambda|^{\alpha-1} + ...\right]^N \to e^{-NA|\lambda|^{\alpha-1}}$$

[5] *Reference Book: First Steps in Random Walks From Tools to Applications J. Klafter and I.M. Sokolov, page 10*

[6] `https://en.wikipedia.org/wiki/Cauchy_distribution`

2. You can find beautiful definition and some result of Polya theorem in here. [7]. If moments are NOT well defined then one can say they find random walk in some distance with infinite probability, so for $0 < \alpha - 1 < 2$ this can happen because moments are infinite.

[7] http://mathworld.wolfram.com/PolyasRandomWalkConstants.html

3. If we take a PDF for time or take time as a continues variable one can obtain desires result. This means that random walker wait on previous point until time of jump receive. This property called waiting time distribution. Now we have mean squared distant or MSD for any distribution $\langle x^2 \rangle = \langle l^2 \rangle \langle n(t) \rangle$, $\langle l^2 \rangle$ is mean of spatially squared steps and $\langle n(t) \rangle$ is mean of number of steps in time t. If we take power law distribution for waiting time like:

$$\psi(t) \propto \frac{\zeta}{\Gamma(1-\zeta)} \frac{\tau^\zeta}{t^{1+\zeta}}$$

Mean number of steps becomes:

$$\langle n(t) \rangle = \frac{1}{\Gamma(1+\zeta)} \frac{t^\zeta}{\tau^\zeta}$$

and we have:

$$\langle x^2 \rangle \propto \langle l^2 \rangle \left(\frac{t}{\tau} \right)^\zeta$$

we have desired result($\zeta \neq 0.5$).